Chapter 4

Physical model

4.1 Model coordinates

4.1.1 Coordinate systems

The coordinate units, used within the program, are either Cartesian (x, y, z) or spherical (λ, ϕ, z), with the z-axis directed upwards along the vertical. The Cartesian coordinates are defined in a horizontal plane tangent at a location on the Earth's surface. In the spherical system λ and ϕ represent respectively the longitude (positive in the eastern, negative in the western hemisphere) and the latitude (positive in the northern, negative in the southern hemisphere). The vertical coordinate is chosen such that the surface $z = 0$ corresponds to the mean sea water level. This gives

$$z = \zeta(x, y, t) \quad \text{or} \quad z = \zeta(\lambda, \phi, t) \quad \text{at the free surface} \tag{4.1}$$

$$z = -h(x, y) \quad \text{or} \quad z = -h(\lambda, \phi) \quad \text{at the bottom} \tag{4.2}$$

where ζ is the sea surface elevation and h the mean water depth so that the total water depth H is given by $H = h + \zeta$.

Cartesian coordinates make it easier to set up a model grid in the horizontal and can be used for size-limited areas where the Earth's curvature is negligible and the Coriolis frequency can be considered as uniform in space. A further advantage is that the coordinate axes can be arbitrarily rotated in the horizontal. Rotation of a spherical coordinate system is less straightforwarded and is considered by the program as a particular case of a curvilinear grid (see below).

Coordinate units are meters in the Cartesian and decimal degrees in the spherical case with $-180^0 \leq \lambda \leq 180^0$ and $-90^0 \leq \phi \leq 90^0$.

Implementation

Coordinate systems are selected in the model with the switch iopt_grid_sph:

0 : Cartesian

1 : spherical

4.1.2 Coordinate transforms in the horizontal

The program allows to define horizontal grids in a more flexible way through the introduction of curvilinear coordinates. Consider firstly the following general horizontal coordinate transform

$$\xi_1 = f_1(x,y)\,, \quad \xi_2 = f_2(x,y) \tag{4.3}$$

The inverse transform becomes

$$x = F_1(\xi_1,\xi_2)\,, \quad y = F_2(\xi_1,\xi_2) \tag{4.4}$$

The distance between two neighbouring (grid) points is given by

$$\begin{aligned}\Delta d^2 &= \Delta x^2 + \Delta y^2 \\ &= \left[\left(\frac{\partial F_1}{\partial \xi_1}\right)^2 + \left(\frac{\partial F_2}{\partial \xi_1}\right)^2\right]\Delta\xi_1^2 + \left[\left(\frac{\partial F_1}{\partial \xi_2}\right)^2 + \left(\frac{\partial F_2}{\partial \xi_2}\right)^2\right]\Delta\xi_2^2 \\ &+ 2\left(\frac{\partial F_1}{\partial \xi_1}\frac{\partial F_1}{\partial \xi_2} + \frac{\partial F_2}{\partial \xi_1}\frac{\partial F_2}{\partial \xi_2}\right)\Delta\xi_1\Delta\xi_2\end{aligned} \tag{4.5}$$

If

$$\frac{\partial F_1}{\partial \xi_1}\frac{\partial F_1}{\partial \xi_2} + \frac{\partial F_2}{\partial \xi_1}\frac{\partial F_2}{\partial \xi_2} = 0 \tag{4.6}$$

then

$$\Delta d^2 = h_1^2\Delta\xi_1^2 + h_2^2\Delta\xi_2^2 \tag{4.7}$$

and (ξ_1, ξ_2) are then called orthogonal curvilinear coordinates. This means geometrically that the coordinate curve along which ξ_1 is a constant, intersects the curve along which ξ_2 is constant orthogonally.

Note that spherical coordinates can be considered as “pseudo”-curvilinear coordinates with respect to Cartesian coordinates with $h_1 = R\cos\phi$ and $h_2 = R$ where R is the mean radius of the Earth, defined as the radius of a sphere having the same volume as the Earth or 6371 km (see Appendix 2 of Gill, 1982)

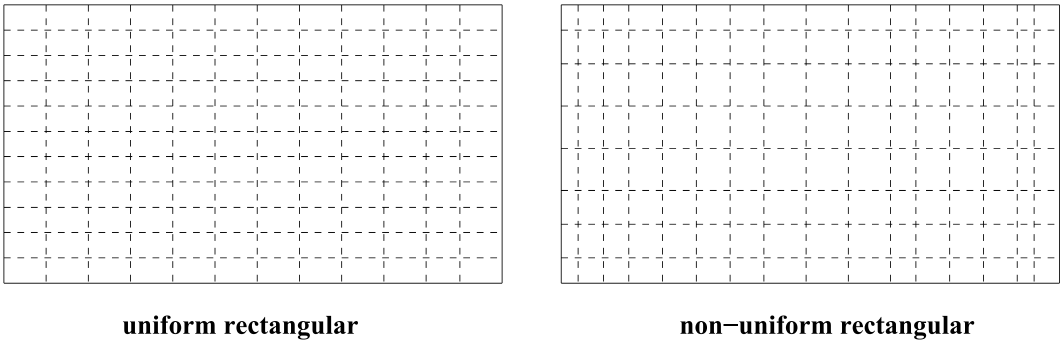

Figure 4.1: Example of a uniform (left) and non-uniform rectangular grid (right).

In practice $\Delta\xi_1, \Delta\xi_2$ are both normalised to 1, so that the metric coefficients h_1, h_2 now become the grid spacings along the curvilinear coordinate lines.

The following types of coordinate transformations are considered in COHERENS:

- "Fully" curvilinear: $h_1 = h_1(\xi_1, \xi_2)$ and $h_2 = h_2(\xi_1, \xi_2)$
- Non-uniform rectangular: $h_1 = h_1(\xi_1)$ and $h_2 = h_2(\xi_2)$, i.e. $\partial h_1/\partial\xi_2 = \partial h_2/\partial\xi_1 = 0$
- Uniform rectangular: h_1 and h_2 are independent of ξ_1 and ξ_2.

A computational model grid in the horizontal is constructed at the "grid nodes" which are located at the orthogonal intersections of a series of coordinate lines along which ξ_1 is constant with coordinate lines along which ξ_2 is constant. The boxes bounded by four neighbouring grid nodes are called model "grid cells". Figures 4.1 and 4.2 show examples of a uniform and non-uniform rectangular grid, repectively a curvilinear grid. Although not recommended, COHERENS offers the possibility to define model grids with "ragged" boundaries, obtained by removing grid cells from the physical domain where the actual calculations are performed. An example is given in Figure 4.3.

A model grid is defined in practice by supplying the coordinates of all grid nodes. In case of a fully curvilinear grid, these coordinates need to be provided by the user. For a rectangular grid, they are given by

$$
\begin{aligned}
x_1 &= x_r \\
x_i &= x_r + \sum_{k=1}^{i-1} \Delta x(k) \quad \text{for} \quad i = 2, \ldots, N_{x+1}
\end{aligned}
$$

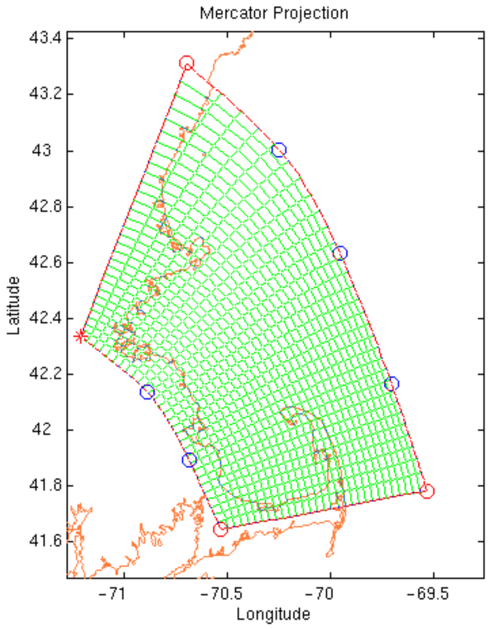

Figure 4.2: Example of a curvilinear grid

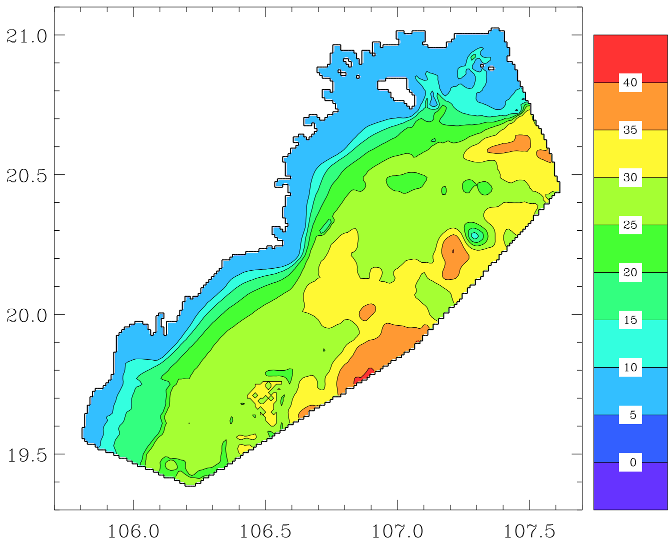

Figure 4.3: Example of a model grid with ragged boundaries

$$
\begin{aligned}
y_1 &= y_r \\
y_j &= y_r + \sum_{k=1}^{j-1} \Delta y(k) \quad \text{for} \quad j = 2, \ldots, N_{y+1}
\end{aligned}
\tag{4.8}
$$

in Cartesian, or

$$
\begin{aligned}
\lambda_1 &= \lambda_r \\
\lambda_i &= \lambda_r + \sum_{k=1}^{i-1} \Delta\lambda(k) \quad \text{for} \quad i = 2, \ldots, N_{x+1} \\
\phi_1 &= \phi_r \\
\phi_j &= \phi_r + \sum_{k=1}^{j-1} \Delta\phi(k) \quad \text{for} \quad j = 2, \ldots, N_{y+1}
\end{aligned}
\tag{4.9}
$$

in spherical coordinates, where N_x, N_y are the number of grid cells in the ξ_1-, respectively ξ_2-direction, (x_r, y_r) or λ_r, ϕ_r the coordinates of a reference point, $(\Delta x, \Delta y)$ the grid spacings along the X- and Y-direction in the Cartesian and $(\Delta\lambda, \Delta\phi)$ the grid spacings along longitude and latitude circles in the spherical case.

Implementation

The type of horizontal grid is selected with the switch iopt_grid_htype:

1: uniform rectangular

2: non-uniform rectangular

3: fully curvilinear

4.1.3 Rotated grids

To avoid that the coordinate lines of a spherical rectangular grid are restrained to latitude and longitude circles, COHERENS allows to apply a grid rotation. This is affected by deplacing the North pole for the geographic coordinates to a new position. In this way the grid can be made more aligned with coastal or more efficient open boundaries. An example of a rotated grid is shown in Figure 4.4

If (λ_p, ϕ_p) are the longitude and latitude of the displaced North Pole, the transformation formulae to the new coordinates (λ', ϕ') become

$$\phi' = \arcsin\Big[\sin\phi_p \sin\phi + \cos\phi_p \cos\phi \cos(\lambda - \lambda_p)\Big] \tag{4.10}$$

$$\lambda' = \mathcal{S}\big(\sin(\lambda_p - \lambda)\big) \arccos\Big[\frac{\cos\phi_p \sin\phi - \sin\phi_p \cos\phi \cos(\lambda - \lambda_p)}{\cos\phi'}\Big] \tag{4.11}$$

where $\mathcal{S}(x)$ is the Sign function[1]. The backward transformation formulae are

$$\phi = \arcsin\Big[\sin\phi_p \sin\phi' + \cos\phi_p \cos\phi' \cos\lambda'\Big] \tag{4.12}$$

$$\lambda = \mathcal{S}(\sin\lambda') \arccos\Big[\frac{\sin\phi_p \cos\phi' \cos\lambda' - \cos\phi_p \sin\phi'}{\cos\phi}\Big] + \lambda_p - \mathcal{S}(\lambda_p)\pi \tag{4.13}$$

The coordinates of the grid nodes in the new coordinate grid are determined by (4.9) with $(\lambda, \phi, \Delta\lambda, \Delta\phi)$ replaced by $(\lambda', \phi', \Delta\lambda', \Delta\phi')$. The location of the new North pole is obtained by defining two additional parameters

- The first is the grid rotation angle α defined as the angle between the geographical and new equator. It is easily seen that $\alpha = 90^0 - \phi_p$. Note that $0 < \alpha < 180^0$.

[1] $\mathcal{S}(x)$ equals 1 if $x \geq 0$, and 0 otherwise.

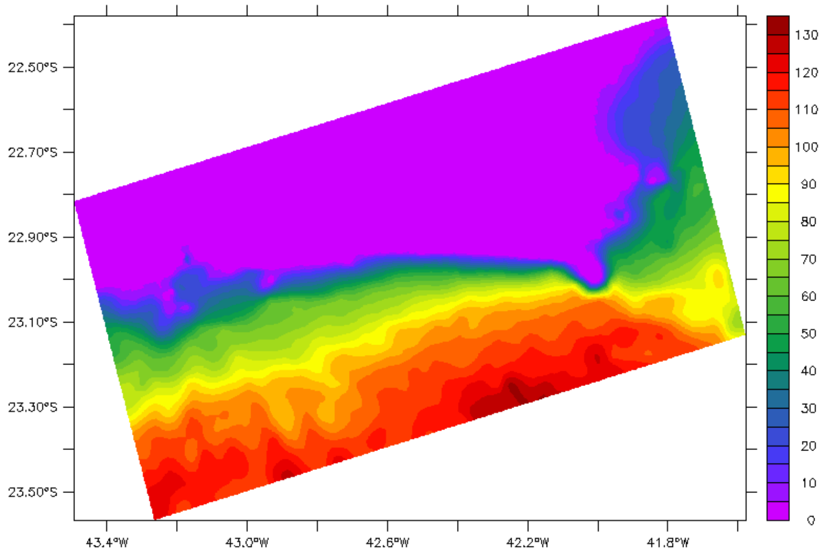

Figure 4.4: Example of a rotated grid

- The second parameter is the reference latitude ϕ'_r from which the longitude of the new North pole can be determined using (4.10):

$$\lambda_p = \lambda_r - \mathcal{S}(\cos\alpha)\left|\arccos\left(\frac{\sin\phi'_r - \cos\alpha\sin\phi_r}{\sin\alpha\cos\phi_r}\right)\right| \tag{4.14}$$

  The reason for taking ϕ'_r as a user-defined parameter is to allow more flexibility for selecting the grid spacing. A uniform rectangular grid with $h_1 \simeq h_2$ can be generated by letting $\Delta\lambda' = \Delta\phi'$ and $\phi'_r = 0$.

A rotated grid in Cartesain coordinates is defined by rotating the axes over the grid angle α and taking the origin of the new Cartesian frame at the reference point (x_r, y_r). The coordinate transformations are given by

$$x' = (x - x_r)\cos\alpha + (y - y_r)\sin\alpha \tag{4.15}$$

$$y' = (y - y_r)\cos\alpha - (x - x_r)\sin\alpha \tag{4.16}$$

and

$$x = x_r + x'\cos\alpha - y'\sin\alpha \tag{4.17}$$

$$y = y_r + y' \cos\alpha + x' \sin\alpha \tag{4.18}$$

4.1.4 Coordinate transforms in the vertical

4.1.4.1 σ-coordinates

The σ-coordinate is defined by (Phillips, 1957)

$$\sigma = \frac{z+h}{H} = \frac{z+h}{h+\zeta} \tag{4.19}$$

where σ varies between 0 at the bottom and 1 at the surface[2]. The reverse formula is obviously

$$z = \sigma H - h \tag{4.20}$$

so that the grid spacing in the vertical becomes

$$\Delta z = H \Delta\sigma \tag{4.21}$$

The spacings of vertical σ-points $\Delta\sigma$ are horizontally uniform, but can be taken as either uniform or non-uniform in the vertical.

Advantages are:

- much simpler boundary conditions at the surface and bottom
- a better resolution of surface and bottom layers

However there are well-known disadvantages of using σ-coordinates:

- areas with steep bathymetric gradients are difficult to present
- large errors can be produced by discretisation of the baroclinic pressure gradient

A non-uniform σ-grid can be obtained by means of a transformation of the form

$$\hat{\sigma} = F(\sigma) \quad \text{or its inverse} \quad \sigma = G(\hat{\sigma}) \tag{4.22}$$

where F and G are increasing functions and $\hat{\sigma}$ equals 0 at the bottom and 1 at the surface. Davies & Jones (1991) defined the following logarithmic transformations

$$\sigma = \frac{1}{\alpha}\left[\ln(1 + \frac{\hat{\sigma}}{\sigma_0}) + \frac{\hat{\sigma}}{\sigma_*}\right] \tag{4.23}$$

[2]Note that the definition is different from the traditional one $\sigma = (z - \zeta)/H$ with $-1 \le \sigma \le 0$.

$$\sigma = 1 - \frac{1}{\alpha}\left[\ln(1 + \frac{1-\hat{\sigma}}{\sigma_0}) + \frac{1-\hat{\sigma}}{\sigma_*}\right] \tag{4.24}$$

where

$$\alpha = \ln(1 + \frac{1}{\sigma_0}) + \frac{1}{\sigma_*} \tag{4.25}$$

The first (second) form provides a more refined resolution at the bottom (surface). The extent of the logarithmic grid is set by the tunable parameter σ_*.

Burchard & Bolding (2002) considered a formulation with refined resolutions near both the bottom and surface

$$\hat{\sigma} = \frac{\tanh\left[(d_l + d_u)\sigma - d_l\right] + \tanh d_l}{\tanh d_l + \tanh d_u} \tag{4.26}$$

Increasing the values of the (positive) parameters d_l or d_u will provide a higher resolution in respectively the bottom or surface layer at the expense of a coarser resolution in the remaining parts of the water column.

A vertical grid is then constructed by firstly taking a series of uniformly spaced σ-levels, i.e. $\sigma_k = (k-1)/N, k = 1, N+1$ where N is the number of vertical layers. In the case of a non-uniform grid, the corresponding values of $\hat{\sigma}_k$ are obtained from the transformation formula. Examples are given in Figure 4.5a-b. The first one shows that the vertical grid positions are more densily packed and the grid spacings are smaller in the bottom (surface) layer for a logarithmic transformation concentrated at the bottom (surface). The Burchard & Bolding (2002) formulation (with $d_l = d_u$) has enhanced resolutions both near the surface as near the bottom but a coarser resolution in the middle of the water column.

4.1.4.2 generalised σ-coordinates

Instead of using the traditional σ-coordinate a generalised vertical "s" coordinate can be defined by

$$z = F(x_1, x_2, s, t) \tag{4.27}$$

with (x_1,x_2) = (x,y), (λ,ϕ) or (ξ_1,ξ_2) and where $s = 0$ at the bottom and $s = 1$ at the surface so that

$$F(x_1, x_2, 0, t) = -h\,, \quad F(x_1, x_2, 1, t) = \zeta \tag{4.28}$$

The vertical grid spacing now becomes

$$\Delta z = \frac{\partial F}{\partial s}\Delta s = h_3 \Delta s \tag{4.29}$$

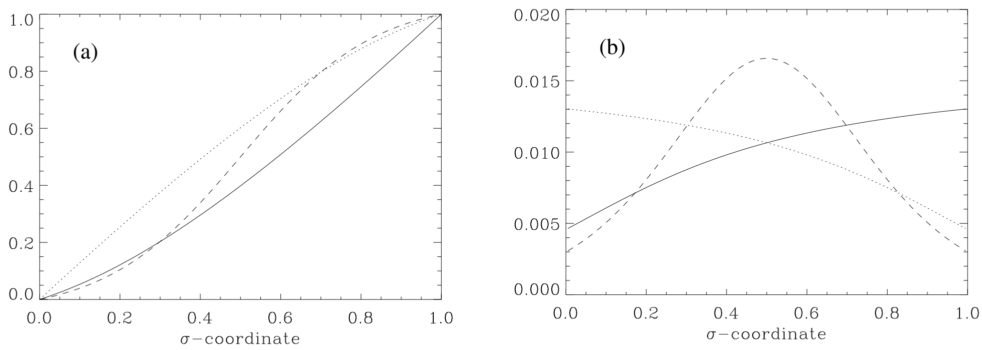

Figure 4.5: Transformed coordinate $\hat{\sigma} = F(\sigma)$ (a) and vertical grid spacing normalised to the total water depth $\Delta\hat{\sigma} = \partial F/\partial\sigma/N$ (b): formulation (4.23) with $\sigma_* = 0.25$, $s_0 = 0.1$ (solid), (4.24) with the same parameter values (dots), (4.26) with $d_l = d_u = 1.5$ (dashes).

The distance between two neighbouring points in 3-D space now becomes

$$\Delta d^2 = h_1^2 \Delta\xi_1^2 + h_2^2 \Delta\xi_2^2 + h_3^2 \Delta s^2 \tag{4.30}$$

Song & Haidvogel (1994) related the s-coordinate to the σ-coordinate by letting

$$F = sH - h + hF_*(x_1, x_2, s)\,, \quad F_*(x_1, x_2, 0) = F_*(x_1, x_2, 1) = 0 \tag{4.31}$$

Equation (4.29) is re-written as

$$h_3 = H + h\frac{\partial F_*}{\partial s} = H(1 + \frac{h}{H}\frac{\partial F_*}{\partial s}) \simeq H(1 + \frac{\partial F_*}{\partial s}) \tag{4.32}$$

where the approximation is made that $h \simeq H$. The assumption is reasonable since the s-coordinate is designed for non-shallow areas with large bathymetric gradients, such as shelf breaks. The s-coordinate, defined by (4.31) is related to the σ-coordinate by

$$\sigma = s + \frac{h}{H}F_*(x_1, x_2, s) \simeq s + F_*(x_1, x_2, s) \tag{4.33}$$

and

$$\Delta\sigma = \frac{\Delta z}{H} \simeq (1 + \frac{\partial F_*}{\partial s})\Delta s \tag{4.34}$$

which means that the Sung-Haidvogel s-coordinate can be seen as a generalised σ-coordinate with non-uniform spacings in the horizontal if $F_* \neq 0$.

The new coordinate should be defined so that it can represent surface and bottom layers in shallow as well as deep waters and can deal with areas with a steep topography. Song & Haidvogel (1994) proposed the following expression for $F_*(s)$:

$$F_*(x_1, x_2, s, t) = \max\left[0, \frac{h - h_c}{h}(C(s) + 1 - s)\right]$$
$$C(s) = \frac{(1-b)\sinh[\theta(s-1)]}{\sinh\theta} + \frac{b}{2}\left[\frac{\tanh[\theta(s-0.5)]}{\tanh(0.5\theta)} - 1\right] \qquad (4.35)$$

where h_c is a critical water depth below which the s-coordinate reduces to the σ-coordinate and b and θ are tunable parameters. The vertical grid is defined by taking uniformly spaced s-levels, i.e. $s_k = (k-1)/N, k = 1, N+1$ and calculating the corresponding generalised σ-levels using (4.33). Figure 4.6

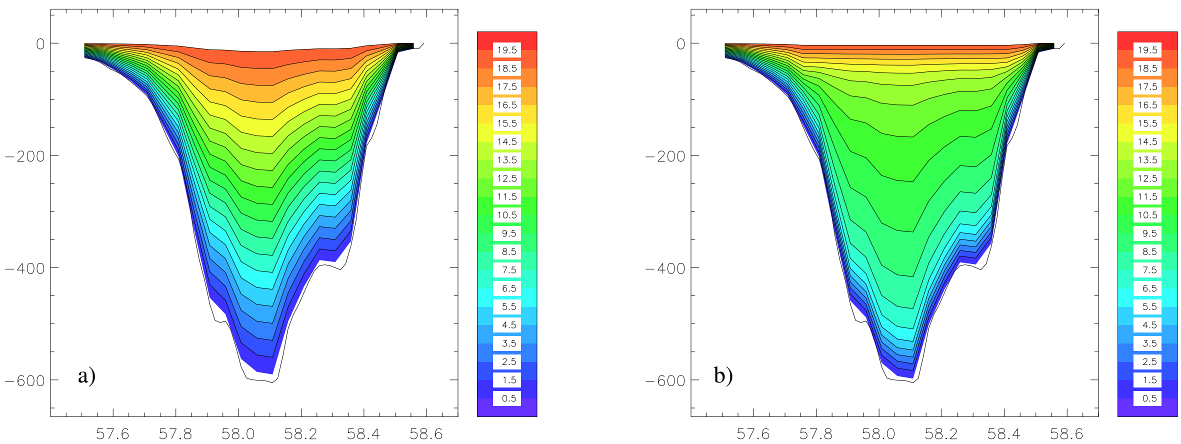

Figure 4.6: Distribution of vertical levels along a transect from Denmark to Norway: uniform σ-coordinates (a), non-uniform σ-coordinates using (4.35) (b).

compares the distribution of vertical levels for a uniform σ-spacing with the s-coordinate levels obtained from (4.35), using $b = 1$, $h_c = 200$ and $\theta = 8$, for a transect across the Norwegian trench in the North Sea. While the σ-coordinate provides an accurate resolution in the shallow waters near the Danish coast (left side of the figure), the layer thickness is $\sim$30 m in the deepest part which is clearly insufficient. A (vertically) non-uniform σ-grid will not resolve the problem since any improvement for the deepest parts will detoriate the solution in the coastal areas. The s-coordinate has a much more accurate resolution in deep water as seen in the figure on the right and reduces to the σ-coordinate when $h < h_c$ near the coasts.

A transformed vertical grid can also be constructed through the inverse relation

$$s = G(x_1, x_2, z, t) \tag{4.36}$$

or, after substituting from (4.20)

$$s = G_*(\sigma, x, y, t) \tag{4.37}$$

with $G_*(0, x, y, t) = 0$, $G_*(1, x, y, t) = 1$ and $\partial G_*/\partial\sigma > 0$. For example, Burchard & Bolding (2002) proposed

$$s_k = \alpha\sigma_k + (1 - \alpha)\hat{\sigma}_k \tag{4.38}$$

where

$$\alpha = \min\Big[\frac{(\hat{\sigma}_k - \hat{\sigma}_{k-1}) - (\sigma_k - \sigma_{k-1})h_c/h}{(\hat{\sigma}_k - \hat{\sigma}_{k-1}) - (\sigma_k - \sigma_{k-1})}, 1\Big] \tag{4.39}$$

and $\hat{\sigma}$ is obtained from (4.26) with h_c a critical water depth below which $s = \sigma$.

4.1.4.3 normalised vertical coordinate

In analogy with the horizontal curvilinear coordinate system the vertical s-coordinate is normalised using

$$H\Delta\sigma = h_3\Delta s \tag{4.40}$$

Setting $\Delta s = 1$ between neighbouring grid points in the (transformed) vertical direction and using similar normalised coordinates in the horizontal one has

$$\Delta d^2 = h_1^2\Delta\xi_1^2 + h_2^2\Delta\xi_2^2 + h_3^2\Delta s^2 = h_1^2 + h_2^2 + h_3^2 \tag{4.41}$$

so that h_1, h_2, h_3 become the grid spacings in the three (transformed) coordinate directions.

Implementation

The type of vertical grid is selected with the switch iopt_grid_vtype

1 : uniform σ-grid

2 : non-uniform vertical σ-grid

3 : non-uniform σ in the horizontal and vertical

In the following, the general vertical coordinate will be represented by its normalised value s. This implies that

$$\frac{\partial}{\partial z} = \frac{1}{H}\frac{\partial}{\partial \sigma} = \frac{1}{h_3}\frac{\partial}{\partial s} \tag{4.42}$$

4.2 Basic model equations

4.2.1 3-D mode equations

4.2.1.1 Cartesian coordinates

The model equations are derived with the following (classic) assumptions.

1. The Boussinesq approximation is applied which means that the density is constant except for the Earth's gravity force.

2. The vertical component of the momentum equations reduces to the hydrostatic balance between the vertical pressure gradient and the gravity force.

3. The horizontal component of the Earth's rotation vector is set to zero. The assumption becomes invalid for non-hydrostatic water masses or near the equator.

The equations for the "3-D" mode consist of the continuity equation, the momentum equations and the equations of temperature and salinity. In Cartesian coordinates and using the previous assumptions these are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.43}$$

$$\begin{aligned}\frac{\partial u}{\partial t} &+ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv \\ &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_x^t + \frac{\partial}{\partial z}\Big(\nu_T\frac{\partial u}{\partial z}\Big) + \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{xy}\end{aligned} \tag{4.44}$$

$$\begin{aligned}\frac{\partial v}{\partial t} &+ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu \\ &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F_y^t + \frac{\partial}{\partial z}\Big(\nu_T\frac{\partial v}{\partial z}\Big) + \frac{\partial}{\partial x}\tau_{yx} + \frac{\partial}{\partial y}\tau_{yy}\end{aligned} \tag{4.45}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{4.46}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}$$
$$= \frac{1}{\rho_0 c_p}\frac{\partial I}{\partial z}+\frac{\partial}{\partial z}\Big(\lambda_T\frac{\partial T}{\partial z}\Big)+\frac{\partial}{\partial x}\Big(\lambda_H\frac{\partial T}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(\lambda_H\frac{\partial T}{\partial y}\Big) \quad (4.47)$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}+v\frac{\partial S}{\partial y}+w\frac{\partial S}{\partial z}$$
$$= \frac{\partial}{\partial z}\Big(\lambda_T\frac{\partial S}{\partial z}\Big)+\frac{\partial}{\partial x}\Big(\lambda_H\frac{\partial S}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(\lambda_H\frac{\partial S}{\partial y}\Big) \quad (4.48)$$

where (u,v) are the horizontal components of the current, w the vertical current, f the Coriolis frequency given by $2\Omega\sin\phi$ where $\Omega=\pi/43082$ radians/s is the Earth's rotation frequency, p the pressure, ρ the density, ρ_0 a uniform reference density, g the acceleration of gravity, (F^t_x,F^t_y) the components of the astronomical tidal force, ν_T and λ_T the vertical turbulent diffusion coefficients, τ_{ij} the horizontal friction tensor, T potential temperature, I the solar irradiance within the water column, c_p the specific heat capacity of sea water at constant pressure, and S salinity.

Note that T is not the *in situ* temperature but potential temperature, defined as the temperature of a fluid parcel, moved adiabatically to a certain level (usually taken at or near the surface). The reason is that (4.47) is derived from the conservation equation of heat. This equation contains an extra term due to compresssibility, which vanishes if T is interpreted as potential temperature (Gill, 1982).

Since the model does not allow for the formation of sea ice at the surface, the temperature must stay above the freezing point of seawater, i.e.

$$T>\alpha_f S\,,\quad \alpha_f=-0.0575\ ^0\mathrm{C/PSU} \quad (4.49)$$

The horizontal diffusion tensor is introduced to represent horizontal subgrid scale processes not resolved by the model and is parameterised as follows

$$\tau_{xx}=-\tau_{yy}=\nu_H D_T\,,\quad \tau_{xy}=\tau_{yx}=\nu_H D_S \quad (4.50)$$

$$D_T=\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\qquad D_S=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \quad (4.51)$$

where D_T and D_S are called the horizontal tension and shearing strain and ν_H denotes the horizontal diffusion coefficient which is either given as a constant or taken as proportional to the local rate of strain

$$\nu_H=C_m\Delta x\Delta y\sqrt{D_T^2+D_S^2} \quad (4.52)$$

Similarly, the scalar diffusion coefficient is either a constant or given by

$$\lambda_H = C_s \Delta x \Delta y \sqrt{D_T^2 + D_S^2} \tag{4.53}$$

Equations (4.52) and (4.53) are the well-known Smagorinsky (1963) parameterisations. The coefficients C_m and C_s usually have the same value of the order of 0.1–0.2. The sub-grid parameterisation (4.50) and (4.51) differs from the one implemented in COHERENS V1 and several other ocean models. The present formulation has been introduced in the Modular Ocean Model (Pacanowski & Griffies, 2000; Griffies, 2004) on general considerations about basic symmetry properties of the physical system.

In the present implementation it is assumed that "horizontal" diffusion of momentum and scalars takes place along horizontal planes. Diffusion along the vertical is parameterised by the vertical diffusion coefficients ν_T and λ_T. It is, however, physically more meaningfull to replace these notions of horizontal mixing, produced by two-dimensional meso-scale eddies, and vertical mixing, representing small scale turbulence on scales of 10^{-3} to 10 m, by mixing along and across isopycnals. Since $\nu_T \ll \nu_H$ and $\lambda_T \ll \lambda_H$, the horizontal mixing scheme may produce an excessive diapycnal diffusion in the presence of lateral fronts. Complex schemes for isopycnal mixing have been developed and applied to global ocean models (e.g. Griffies *et al.*, 1998). They are not implemented in the current version of COHERENS. The main reason is that the program is primarily developed for coastal and regional seas where a sufficiently high resolution can be taken to resolve meso-scale eddies in the horizontal.

The pressure can be eliminated from the above equations by rewriting (4.46) as

$$\frac{\partial p}{\partial z} = -\rho_0 (g - b) \tag{4.54}$$

where the buoyancy b is defined by

$$b = -\left(\frac{\rho - \rho_0}{\rho_0}\right) g \tag{4.55}$$

Integrating (4.54) using the surface boundary condition $p = P_a$ at $z = \zeta$ where P_a is the surface atmospheric pressure, the horizontal pressure gradient terms in (4.44) and (4.45) can be written as

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} = -g\frac{\partial \zeta}{\partial x_i} - \frac{1}{\rho_0}\frac{\partial P_a}{\partial x_i} - \frac{\partial q}{\partial x_i} \tag{4.56}$$

where x_i equals x or y and q is the vertically integrated buoyancy

$$q = -\int_z^{\zeta} b\, dz \tag{4.57}$$

The first two terms on the right of (4.56) represent the barotropic, the third one the baroclinic component.

The following additional remarks are to be given:

- The vertical diffusion term and the diffusion coefficients ν_T and λ_T are obtained from a turbulence model. Various schemes, including simple algebraic schemes and more complex second order closure schemes, are implemented (see Section 4.4).

- The acceleration due to gravity can be set in the program to a constant value or obtained from the geodetic formula as function of latitude (see Appendix 2 of Gill, 1982)

$$g = 9.78032 + 0.005172\sin^2\phi - 0.00006\sin^2 2\phi \tag{4.58}$$

- The absorption of solar irradiance within the water column is generally a function of solar wavelength and the penetration depth of solar light. The formulation chosen in the model follows the one given by Paulson & Simpson (1977) whereby I is given by

$$I(x_1, x_2, z) = Q_{rad}\left(Re^{-z/\lambda_1} + (1-R)e^{-z/\lambda_2}\right) \tag{4.59}$$

  where R represents the absorption of the red end of the solar spectrum in the upper (1–2) meters of the water column, $1-R$ the absorption of blue-green light over larger depths and Q_{rad} the solar radiance incident on the surface. Since turbidity effects are not explicitly taken into account by the physical model, values of R, λ_1, $\lambda_2 \gg \lambda_1$ depend on the optical properties of the water masses and can be selected following e.g. the optical classification scheme of Jerlov (1968). Solar radiance is further discussed in Section 4.6.

- The astronomical force is only relevant in deep ocean waters and can be neglected on the shelf and in the coastal zone. Expressions for its components are given as a sum of tidal harmonics (see Section 4.5).

4.2.1.2 transformed coordinates

The forms of the model equations in orthogonal curvilinear and s-coordinates (ξ_1,ξ_2,s) are derived in Appendix A. The equations of continuity and momentum, written in conservative and operator format, become

$$\frac{1}{h_3}\frac{\partial h_3}{\partial t} + \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial \xi_1}(h_2h_3u) + \frac{\partial}{\partial \xi_2}(h_1h_3v)\right] + \frac{1}{h_3}\frac{\partial \omega}{\partial s} = 0 \tag{4.60}$$

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 u) + \mathcal{A}_{h1}(u) + \mathcal{A}_{h2}(u) + \mathcal{A}_v(u) + \frac{v}{h_1 h_2}\left(u\frac{\partial h_1}{\partial \xi_2} - v\frac{\partial h_2}{\partial \xi_1}\right) - 2\Omega v \sin\phi$$
$$= -\frac{g}{h_1}\frac{\partial \zeta}{\partial \xi_1} - \frac{1}{\rho_0 h_1}\frac{\partial P_a}{\partial \xi_1} + F_1^b + F_1^t + \mathcal{D}_{mv}(u) + \mathcal{D}_{mh1}(\tau_{11}) + \mathcal{D}_{mh2}(\tau_{12}) \tag{4.61}$$

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 v) + \mathcal{A}_{h1}(v) + \mathcal{A}_{h2}(v) + \mathcal{A}_v(v) + \frac{u}{h_1 h_2}\left(v\frac{\partial h_2}{\partial \xi_1} - u\frac{\partial h_1}{\partial \xi_2}\right) + 2\Omega u \sin\phi$$
$$= -\frac{g}{h_2}\frac{\partial \zeta}{\partial \xi_2} - \frac{1}{\rho_0 h_2}\frac{\partial P_a}{\partial \xi_2} + F_2^b + F_2^t + \mathcal{D}_{mv}(v) + \mathcal{D}_{mh1}(\tau_{21}) + \mathcal{D}_{mh2}(\tau_{22}) \tag{4.62}$$

where ω represents the transformed vertical velocity, further discussed below. The spherical case is recovered from the above equations by letting $h_1 = R\cos\phi$ and $h_2 = R$.

Apart from a constant factor of proportionality, which can be set to 1 without loss of generality, the metric coefficients h_i are related to the model grid spacings along the horizontal coordinate directions (Δx,Δy) and the vertical (Δz) by

$$\Delta x = h_1\,, \quad \Delta y = h_2, \quad \Delta z = h_3 \Delta s \tag{4.63}$$

The advection and diffusion operators are defined by

$$\mathcal{A}_{h1}(F) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial \xi_1}(h_2 h_3 u F) \tag{4.64}$$
$$\mathcal{A}_{h2}(F) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial \xi_2}(h_1 h_3 v F) \tag{4.65}$$
$$\mathcal{A}_v(F) = \frac{1}{h_3}\frac{\partial}{\partial s}(\omega F) \tag{4.66}$$

$$\mathcal{D}_{mh1}(F) = \frac{1}{h_1 h_2^2 h_3}\frac{\partial}{\partial \xi_1}(h_2^2 h_3 F) \tag{4.67}$$
$$\mathcal{D}_{mh2}(F) = \frac{1}{h_1^2 h_2 h_3}\frac{\partial}{\partial \xi_2}(h_1^2 h_3 F) \tag{4.68}$$
$$\mathcal{D}_{mv}(F) = \frac{1}{h_3}\frac{\partial}{\partial s}\left(\frac{\nu_T}{h_3}\frac{\partial F}{\partial s}\right) \tag{4.69}$$

The parameterised form of the horizontal shear stress tensor in orthogonal curvilinear coordinates is given by Pacanowski & Griffies (2000); Griffies (2004)

$$\tau_{11} = -\tau_{22} = \nu_H D_T\,, \quad \tau_{12} = \tau_{21} = \nu_H D_S \tag{4.70}$$

$$\begin{aligned} D_T &= \frac{h_2}{h_1}\frac{\partial}{\partial \xi_1}\left(\frac{u}{h_2}\right) - \frac{h_1}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{v}{h_1}\right) \\ D_S &= \frac{h_1}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{u}{h_1}\right) + \frac{h_2}{h_1}\frac{\partial}{\partial \xi_1}\left(\frac{v}{h_2}\right) \end{aligned} \tag{4.71}$$

The quantity ω in (4.60) and (4.66) is the transformed vertical current normal to the s-coordinate surfaces. Physical and transformed vertical current are related by

$$w = \omega - h_3\left(\frac{\partial s}{\partial t} + \frac{u}{h_1}\frac{\partial s}{\partial \xi_1} + \frac{v}{h_2}\frac{\partial s}{\partial \xi_2}\right) \tag{4.72}$$

An alternative form, more useful for numerical discretisation, is

$$w = \frac{1}{h_3}\frac{\partial}{\partial t}(h_3 z) + \mathcal{A}_{h1}(z) + \mathcal{A}_{h2}(z) + \mathcal{A}_v(z) \tag{4.73}$$

The horizontal vector F_i^t represents the components of the astronomical tidal force, discussed in Section 4.5. The expression for the baroclinic pressure gradient now contains two terms as a consequence of the vertical coordinate transformation

$$F_i^b = -\frac{1}{h_i h_3}\left[\frac{\partial}{\partial \xi_i}(h_3 q) - \frac{\partial}{\partial s}\left(q\frac{\partial z}{\partial \xi_i}\right)\right] \tag{4.74}$$

where

$$q = -\int_s^1 b h_3\, ds \tag{4.75}$$

The equations for potential temperature and salinity can, in the absence of a particle sinking term, be cast in a more general form, representing the transport of an arbitrary concentration ψ (T, S, sediment, contaminant, biological state variable)

$$\begin{aligned} \frac{1}{h_3}\frac{\partial}{\partial t}(h_3\psi) &+ \mathcal{A}_{h1}(\psi) + \mathcal{A}_{h2}(\psi) + \mathcal{A}_v(\psi) \\ &= \mathcal{P}(\psi) - \mathcal{S}(\psi) + \mathcal{D}_{sv}(\psi) + \mathcal{D}_{sh1}(\psi) + \mathcal{D}_{sh2}(\psi) \end{aligned} \tag{4.76}$$

where $\mathcal{P}(\psi)$, $\mathcal{S}(\psi)$ represent all source, respectively sinks terms. Two-dimensional diffusion of scalars is taken along constant s-surfaces. This gives

$$\mathcal{D}_{sh1}(\psi) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial \xi_1}\left(\lambda_H \frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial \xi_1}\right) \tag{4.77}$$

$$\mathcal{D}_{sh2}(\psi) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial \xi_2}\left(\lambda_H \frac{h_1 h_3}{h_2}\frac{\partial \psi}{\partial \xi_2}\right) \tag{4.78}$$

$$\mathcal{D}_{sv}(\psi) \quad = \quad \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\lambda_T^{\psi}}{h_3}\frac{\partial F}{\partial s}\Big) \tag{4.79}$$

where λ_T^{ψ} is the vertical diffusion coefficient for the scalar ψ[3].

Smagorinsky's diffusion coefficients in curvilinear coordinates are given by

$$\nu_H \quad = \quad C_m h_1 h_2 \sqrt{D_T^2 + D_S^2} \tag{4.80}$$

$$\lambda_H \quad = \quad C_s h_1 h_2 \sqrt{D_T^2 + D_S^2} \tag{4.81}$$

Applying (4.76) for temperature and salinity one has

$$\begin{aligned} \frac{1}{h_3}\frac{\partial}{\partial t}(h_3 T) \quad &+ \quad \mathcal{A}_{h1}(T) + \mathcal{A}_{h2}(T) + \mathcal{A}_v(T) \\ &= \quad \frac{1}{\rho_0 c_p h_3}\frac{\partial I}{\partial s} + \mathcal{D}_{sv}(T) + \mathcal{D}_{sh1}(T) + \mathcal{D}_{sh2}(T) \end{aligned} \tag{4.82}$$

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 S) + \mathcal{A}_{h1}(S) + \mathcal{A}_{h2}(S) + \mathcal{A}_v(S) = \mathcal{D}_{sv}(S) + \mathcal{D}_{sh1}(S) + \mathcal{D}_{sh2}(S) \tag{4.83}$$

4.2.2 2-D mode equations

The 3-D continuity and momentum equations presented in the previous section need to be supplemented by additional 2-D equations for the depth-integrated current and surface elevation. There are two main reasons.

1. The surface elevation appearing in the momentum equations cannot be determined from the 3-D equations only.

2. The numerical solution of the 3-D equations are constrained by the CFL limit which poses a severe limit on the time step used in the numerical discretisations. This can be resolved by solving the simpler 2-D equations with a smaller 2-D time step and inserting the results into the 3-D equations which can now be solved with a larger 3-D time step. The method, known as the mode splitting technique, is further discussed in Chapter 5.

[3] The same vertical diffusion coefficient is taken for temperature and salinity, i.e. $\lambda_T^T = \lambda_T^S = \lambda_T$.

The 2-D mode equations consists of three equations for the surface elevation and the depth-integrated currents, defined by

$$(U,V) = \int_0^1 (u,v) h_3 \, \mathrm{d}s \tag{4.84}$$

The equations are then obtained by integrating (4.60)–(4.62) over the vertical. This gives

$$\frac{\partial \zeta}{\partial t} + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi_1}(h_2 U) + \frac{\partial}{\partial \xi_2}(h_1 V)\right] = 0 \tag{4.85}$$

$$\begin{aligned}\frac{\partial U}{\partial t} &+ \overline{\mathcal{A}}_{h1}(U) + \overline{\mathcal{A}}_{h2}(U) + \frac{V}{H h_1 h_2}\left(\frac{\partial h_1}{\partial \xi_2}U - \frac{\partial h_2}{\partial \xi_1}V\right) - 2\Omega V \sin\phi \\ &= -\frac{gH}{h_1}\frac{\partial \zeta}{\partial \xi_1} - \frac{H}{\rho_0 h_1}\frac{\partial P_a}{\partial \xi_1} + \overline{F_1^b} + HF_1^t + \tau_{s1} - \tau_{b1} \\ &+ \overline{\mathcal{D}}_{mh1}(\overline{\tau_{11}}) + \overline{\mathcal{D}}_{mh2}(\overline{\tau_{12}}) - \overline{\delta A}_{h1} + \overline{\delta D}_{h1}\end{aligned} \tag{4.86}$$

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \overline{\mathcal{A}}_{h1}(V) + \overline{\mathcal{A}}_{h2}(V) + \frac{U}{H h_1 h_2}\left(\frac{\partial h_2}{\partial \xi_1}V - \frac{\partial h_1}{\partial \xi_2}U\right) + 2\Omega U \sin\phi \\ &= -\frac{gH}{h_2}\frac{\partial \zeta}{\partial \xi_2} - \frac{H}{\rho_0 h_2}\frac{\partial P_a}{\partial \xi_2} + \overline{F_2^b} + HF_2^t + \tau_{s2} - \tau_{b2} \\ &+ \overline{\mathcal{D}}_{mh1}(\overline{\tau_{21}}) + \overline{\mathcal{D}}_{mh2}(\overline{\tau_{22}}) - \overline{\delta A}_{h2} + \overline{\delta D}_{h2}\end{aligned} \tag{4.87}$$

where $\overline{F_i^b}$ are the depth-integrated components of the baroclinic pressure gradient F_i^b, τ_{si} and τ_{bi} are the components of the surface and bottom stress (normalised with the reference density ρ_0)

$$(\tau_{s1}, \tau_{s2}) = \frac{\nu_T}{h_3}\frac{\partial (u,v)}{\partial s}\bigg|_{sur} \tag{4.88}$$

$$(\tau_{b1}, \tau_{b2}) = \frac{\nu_T}{h_3}\frac{\partial (u,v)}{\partial s}\bigg|_{bot} \tag{4.89}$$

The advection and diffusion operators for transports are given by

$$\overline{\mathcal{A}}_{h1}(F) = \frac{1}{h_1 h_2}\frac{\partial}{\partial \xi_1}\left(h_2 \frac{UF}{H}\right) \tag{4.90}$$

$$\overline{\mathcal{A}}_{h2}(F) = \frac{1}{h_1 h_2}\frac{\partial}{\partial \xi_2}\left(h_1 \frac{VF}{H}\right) \tag{4.91}$$

$$\overline{\mathcal{D}}_{mh1} = \frac{1}{h_1 h_2^2} \frac{\partial}{\partial \xi_1} \left(h_2^2 F \right) \tag{4.92}$$

$$\overline{\mathcal{D}}_{mh2} = \frac{1}{h_1^2 h_2} \frac{\partial}{\partial \xi_2} \left(h_1^2 F \right) \tag{4.93}$$

The 2-D equivalents of the shear stress tensor can be written as

$$\overline{\tau_{11}} = -\overline{\tau_{22}} = \overline{\nu_H}\,\overline{D}_T\,, \quad \overline{\tau_{12}} = \overline{\tau_{21}} = \overline{\nu_H}\,\overline{D}_S \tag{4.94}$$

$$\overline{\nu_H} = \int_0^1 \nu_H h_3 \, ds \tag{4.95}$$

$$\begin{aligned} \overline{D}_T &= \frac{h_2}{h_1} \frac{\partial}{\partial \xi_1} \left(\frac{U}{H h_2} \right) - \frac{h_1}{h_2} \frac{\partial}{\partial \xi_2} \left(\frac{V}{H h_1} \right) \\ \overline{D}_S &= \frac{h_1}{h_2} \frac{\partial}{\partial \xi_2} \left(\frac{U}{H h_1} \right) + \frac{h_2}{h_1} \frac{\partial}{\partial \xi_1} \left(\frac{V}{H h_2} \right) \end{aligned} \tag{4.96}$$

The last two terms on the right of (4.86)–(4.87) only contain the baroclinic part of the 3-D current

$$(\delta u, \delta v) = \left(u - \frac{U}{H}, v - \frac{V}{H} \right) \tag{4.97}$$

The explicit forms are

$$\begin{aligned} \overline{\delta A}_{h1} &= \frac{1}{h_1 h_2} \int_0^1 \left[\frac{\partial}{\partial \xi_1} \left(h_2 h_3 \delta u^2 \right) + \frac{\partial}{\partial \xi_2} \left(h_1 h_3 \delta u \delta v \right) \right. \\ &+ \left. h_3 \frac{\partial h_1}{\partial \xi_2} \delta u \delta v - h_3 \frac{\partial h_2}{\partial \xi_1} \delta v^2 \right] ds \end{aligned} \tag{4.98}$$

$$\begin{aligned} \overline{\delta A}_{h2} &= \frac{1}{h_1 h_2} \int_0^1 \left[\frac{\partial}{\partial \xi_1} \left(h_2 h_3 \delta u \delta v \right) + \frac{\partial}{\partial \xi_2} \left(h_1 h_3 \delta v^2 \right) \right. \\ &+ \left. h_3 \frac{\partial h_2}{\partial \xi_1} \delta u \delta v - h_3 \frac{\partial h_1}{\partial \xi_2} \delta u^2 \right] ds \end{aligned} \tag{4.99}$$

$$\overline{\delta D}_{h1} = \frac{1}{h_1 h_2} \left[\frac{1}{h_2} \frac{\partial}{\partial \xi_1} \left(h_2^2 \int_0^1 \nu_H \delta D_T h_3 \, ds \right) + \frac{1}{h_1} \frac{\partial}{\partial \xi_2} \left(h_1^2 \int_0^1 \nu_H \delta D_S h_3 \, ds \right) \right]$$

$$(4.100)$$

$$\overline{\delta D}_{h2} = \frac{1}{h_1 h_2}\left[\frac{1}{h_2}\frac{\partial}{\partial \xi_1}\left(h_2^2 \int_0^1 \nu_H \delta D_S h_3\, ds\right) - \frac{1}{h_1}\frac{\partial}{\partial \xi_2}\left(h_1^2 \int_0^1 \nu_H \delta D_T h_3\, ds\right)\right] \tag{4.101}$$

where δD_T and δD_S are given by (4.71) with (u,v) replaced by $(\delta u,\delta v)$.

The 3-D and 2-D continuity equations involve either the 3-D and 2-D current. An alternative form, involving the baroclinic and depth-integrated currents, can be derived by multiplying (4.60) by h_3 and substracting (4.85) multiplied by the transformed grid spacing h_3/H. This gives

$$\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial \xi_1}(h_2 h_3 \delta u) + \frac{\partial}{\partial \xi_2}(h_1 h_3 \delta v)\right] + \frac{U}{h_1}\frac{\partial}{\partial \xi_1}\left(\frac{h_3}{H}\right) + \frac{V}{h_2}\frac{\partial}{\partial \xi_2}\left(\frac{h_3}{H}\right) + \frac{\partial \omega}{\partial s} = 0 \tag{4.102}$$

The second and third term only arise if the transformed grid spacing varies in the horizontal, i.e. in case of the most general vertical coordinate transformation.

Implementation

The solution method of the previous set of equations is controlled by the following model switches

iopt_mode_2D Switch to enable (1) or disable (0) the solution of the 2-D mode equations. It is advised not to switch off the 2-D mode unless for 1-D water column applications (see Section 4.3.1 below) or for carefully designed applications.

iopt_mode_3D Switch to enable (1) or disable (0) the solution of the 3-D hydrodynamic equations (4.60)–(4.62). It is recommended not to disable the 3-D mode unless for purely 2-D (depth-averaged) applications (in which case the switch is automatically disabled by the program).

iopt_temp Type of update for the temperature field.

- 0 : Temperature is uniform in space and time.
- 1 : Temperature is uniform in time, but non-uniform in space.
- 2 : Temperature is non-uniform in space and time and obtained by solving equation (4.82).

iopt_sal Type of update for the salinity field.

- 0 : Salinity is uniform in space and time.

1 : Salinity is uniform in time, but non-uniform in space.

2 : Salinity is non-uniform in space and time and obtained by solving equation (4.83).

4.2.3 Equation of state

Equations (4.61), (4.62), (4.102), (4.85)–(4.87) and (4.82)–(4.83) form a complete set of equations for u, v, ω, ζ, $\overline{U}$, $\overline{V}$, T and S with the constraints (4.84), provided that the density ρ, which enter the equations through the baroclinic gradient and the turbulent diffusion coefficients ν_T, λ_T (see Section 4.4 below), is known. Contrary to temperature and salinity, the density is not obtained by an additional transport equation but by means of an equation of state (EOS).

The International EOS (Millero *et al.*, 1980), adopted in the previous version of COHERENS relates the density to the three state variables T, S and p where T is the *in situ* temperature. A more appropriate formulation still based on the International EOS, but with *in situ* temperature replaced by potential temperature was considered by Jacket & McDougall (1995). More recently, McDougall *et al.* (2003) proposed an EOS using potential temperature, which according to the authors is more accurate than the International EOS and computationally more efficient. The latter formulation has therefore been implemented in COHERENS V2.0.

With a precision of 0.003 kg/m^3 the density is given by

$$\rho(S,T,p) = P_1(S,T,p)/P_2(S,T,p) \tag{4.103}$$

$$\begin{aligned}
P_1 = a_0 &+ a_1T + a_2T^2 + a_3T^3 + a_4S + a_5ST + a_6S^2 \\
&+ a_7p + a_8pT^2 + a_9pS + a_{10}p^2 + a_{11}p^2T^2 \\
P_2 = 1 &+ b_1T + b_2T^2 + b_3T^3 + b_4T^4 + b_5S + b_6ST + b_7ST^3 \\
&+ b_8S^{3/2} + b_9S^{3/2}T^2 + b_{10}p + b_{11}p^2T^3 + b_{12}p^3T
\end{aligned} \tag{4.104}$$

where a_i and b_i are empirical parameters listed in Table 4.1. Neglecting density variations in the water column and atmospheric pressure, p can be approximated by

$$p \simeq -\rho g(z-\zeta) \tag{4.105}$$

The expansion coefficients for temperature and salinity are obtained from (4.103)–(4.104)

$$\beta_T = -\frac{1}{\rho}\frac{\partial\rho}{\partial T} = \frac{1}{P_2}\frac{\partial P_2}{\partial T} - \frac{1}{P_1}\frac{\partial P_1}{\partial T} \tag{4.106}$$

Table 4.1: Values of the empirical parameters in the McDougall *et al.* (2003) equation of state.

a_0	999.843699	b_1	7.28606739×10^{-3}
a_1	7.3521284	b_2	-4.60835542×10^{-5}
a_2	-5.45928211×10^{-2}	b_3	3.68390573×10^{-7}
a_3	3.98476704×10^{-4}	b_4	1.80809186×10^{-10}
a_4	2.96938239	b_5	2.14691708×10^{-3}
a_5	-7.23268813×10^{-3}	b_6	-9.27062484×10^{-6}
a_6	2.12382341×10^{-3}	b_7	$-1.78343643\times10^{-10}$
a_7	1.04004591×10^{-2}	b_8	4.76534122×10^{-6}
a_8	1.03970529×10^{-7}	b_9	1.63410736×10^{-9}
a_9	5.1876188×10^{-6}	b_{10}	5.30848875×10^{-6}
a_{10}	-3.24041825×10^{-8}	b_{11}	$-3.03175128\times10^{-16}$
a_{11}	-1.2386936×10^{-11}	b_{12}	$-1.27934137\times10^{-17}$

$$\beta_S = \frac{1}{\rho}\frac{\partial\rho}{\partial S} = \frac{1}{P_1}\frac{\partial P_1}{\partial S} - \frac{1}{P_2}\frac{\partial P_2}{\partial S} \tag{4.107}$$

For compatibility with the previous COHERENS version and simple case studies, the model allows to use a simpler linear equations of state, obtained by expanding (4.103)–(4.107) around a reference state

$$\rho = \rho_0\left(1 + \beta_S(S - S_r) - \beta_T(T - T_r)\right) \tag{4.108}$$

where T_r and S_r are constant reference values, and (ρ_0,β_T,β_S) are obtained from (4.103)–(4.104) with $T = T_r$, $S = S_r$, $p = 0$.

Implementation

The following switches, used for the evaluation of density and density gradients, are available:

iopt_dens Selects type of equation of state.

- 0 : The density is set to a uniform value, obtained from (4.103)-(4.104) using constant reference values for T, S and $p = 0$. The expansion coefficients are to zero.
- 1 : Density is calculated from the linear EOS (4.108). Constant values are taken for the expansion coefficients.
- 2 : ρ, β_T and β_S are calculated from (4.103)–(4.104) with $p = 0$.
- 3 : ρ, β_T and β_S are calculated from (4.103)–(4.104) with a non-zero pressure.

iopt_dens_grad Selects the numerical algorithm for discretisation of the baroclinic pressure gradient (see Section 5.3.13 for details).

0 : The gradient is set to zero in all momentum equations.
1 : Traditional σ-coordinate (second order) method.
2 : Using the z-level method.
3 : The method of Shchepetkin & McWilliams (2003)

4.3 Model equations on reduced grids

4.3.1 Water column (1-D) mode

In case of a water column application the horizontal grid reduces to one singular point so that the grid becomes one-dimensional. The following simplifications are made:

1. Advective and horizontal diffusion terms are set to zero.
2. All components of the horizontal pressure gradient are neglected except for the barotropic surface slope term.
3. The continuity equation is not solved. This means in particular that the vertical current is no longer calculated.

In the absence of an horizontal grid, the model equations can be written using Cartesian coordinates in the horizontal and σ-coordinates in the vertical. The momentum equations (4.61), (4.62) then reduce to

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3u) - 2fv = -g\frac{\partial\zeta}{\partial x} + F_1^t + \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\nu_T}{h_3}\frac{\partial u}{\partial s}\Big) \tag{4.109}$$

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3v) + 2fu = -g\frac{\partial\zeta}{\partial y} + F_2^t + \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\nu_T}{h_3}\frac{\partial v}{\partial s}\Big) \tag{4.110}$$

where

- the Coriolis frequency is defined by specifying the latitude of the location, i.e. $f = 2\Omega\sin(\phi_{ref})$
- the surface slope and the surface elevation ζ, needed to calculate the total water depth H and the vertical grid spacing h_3 are specified as external "surface" forcing conditions

The one-dimensional version of the scalar transport equation (4.76) is given by

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3\psi) = \mathcal{P}(\psi) - \mathcal{S}(\psi) + \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\lambda_T^\psi}{h_3}\frac{\partial\psi}{\partial s}\Big) \tag{4.111}$$

4.3.2 Depth-averaged (2-D) mode

In depth-averaged mode it is assumed that the 3-D current and all 3-D scalar quantities are depth-independent.

From (4.102) it follows that $\omega = 0$. The hydrodynamic equations are then given by (4.85)–(4.87) without the baroclinic terms, i.e.

$$\overline{\delta A}_{h1} = \overline{\delta A}_{h2} = \overline{\delta D}_{h1} = \overline{\delta D}_{h2} = 0 \tag{4.112}$$

The depth-integrated form of the transport equation for a depth-independent scalar ψ is obtained by integrating (4.76) over the vertical

$$\begin{aligned}\frac{\partial}{\partial t}\Big(H\overline{\psi}\Big) &+ \overline{\mathcal{A}}_{h1}(H\overline{\psi}) + \overline{\mathcal{A}}_{h2}(H\overline{\psi}) = H\Big(\mathcal{P}(\overline{\psi}) - \mathcal{S}(\overline{\psi})\Big) + F_s^{\psi} - F_b^{\psi} \\ &+ \frac{1}{h_1 h_2}\Big[\frac{\partial}{\partial \xi_1}\Big(\overline{\lambda_H}\frac{h_2}{h_1}\frac{\partial\overline{\psi}}{\partial \xi_1}\Big) + \frac{\partial}{\partial \xi_2}\Big(\overline{\lambda_H}\frac{h_1}{h_2}\frac{\partial\overline{\psi}}{\partial \xi_2}\Big)\Big]\end{aligned} \tag{4.113}$$

where F_s^{ψ} and F_b^{ψ} are the fluxes at respectively the surface and the bottom.

The vertically integrated horizontal diffusion coefficients take the form

$$\overline{\nu_H} = C_m h_1 h_2 H\sqrt{\overline{D}_T^2 + \overline{D}_S^2}\,, \quad \overline{\lambda_H} = C_s h_1 h_2 H\sqrt{\overline{D}_T^2 + \overline{D}_S^2} \tag{4.114}$$

where $\overline{D}_T$, $\overline{D}_S$ are defined by (4.96).

4.4 Turbulence schemes

4.4.1 Introduction

The objective of a turbulence scheme is to parameterise the effects of turbulent motions. It is assumed that turbulence is fully developed and in a quasi-equilibrium state. The main characteristics can be described as follows (see e.g. Ferziger, 2005; Kantha & Clayson, 2000a):

- Three-dimensional. In contrast to the mean flow, which may be two-dimensional, turbulent motions are fully three-dimensional. This definition excludes the two-dimensional turbulence of geophysical flows on the meso-scale, mentioned in Section 4.2 which is parameterised by a horizontal mixing scheme.

- Randomness. Turbulence has a "short-time" memory. This means that turbulent states arising from slight changes in initial, boundary or forcing conditions become uncorrelated in time within short time intervals. As a consequence, the only meaningfull way to analyse turbulence is through its statistical properties.

- Broad spectrum. Turbulent motions span a large spectrum of scales in space and time.
- Vorticity. In contrast to waves, turbulent motions are characterised by random vorticity fluctuations. This explains why turbulence can be visualised in laboratory experiments as a spectrum of eddies on different spatial scales. On the largest scales, the eddies extract energy from the mean flow, whereas on the shortest (so-called Kolmogorov) scales this energy is converted into heat by molecular disssipation.

The spatial scales of turbulence, which need to be taken into account in models for the ocean, shelf seas or coastal areas, range from 10^{-3}–10^2 m and are, except eventually for the largest ones, not resolved by the model. Turbulence schemes need to be developed, based upon the statistical properties of the turbulence spectrum. Starting point are the Navier-Stokes equations and the equations of continuity and temperature[4], written for convenience in Cartesian coordinates and tensorial notation

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} + \epsilon_{ijk}f_jU_k = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} + \delta_{i3}b + \nu\sum_j\frac{\partial^2 U_i}{\partial x_j^2} \tag{4.115}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{4.116}$$

$$\frac{\partial T}{\partial t} + U_i\frac{\partial T}{\partial x_i} = \mathcal{S}(T) + k_T\sum_i\frac{\partial^2 T}{\partial x_i^2} \tag{4.117}$$

where summation is performed within each term over repeating indices, U_i are the velocity components, ϵ_{ijk} is 1(-1) if (i,j,k) are in cyclic (anticyclic) order and 0 if any two indices are equal, δ_{ij} is the Kronecker symbol (1 if i=j, 0 otherwise), ν and k_T are the kinematic viscosity and molecular diffusivity of heat, P the dynamic pressure[5] and $f_i = 2\Omega(\cos\phi, 0, \sin\phi)$ is twice the Earths's rotation vector.

All quantities are then decomposed into a mean (designated by an overbar) and a fluctuating turbulent part or

$$U_i = \overline{U_i} + u_i\,, \quad T = \overline{T} + \theta\,, \quad P = \overline{P} + \rho_0\pi\,, \quad b = \overline{b} + \beta \tag{4.118}$$

where the fluctuating parts have zero means. The averages can be considered as ensemble averages over a large number of turbulent states or as a statistical

[4]In the absence of double-diffusive mixing, which is not implemented in the current model code, the procedure is similar if temperature is replaced by salinity.

[5]The dynamic pressure is defined as the pressure minus its homogeneous hydrostatic part, i.e. $P = p + \rho_0 g(z - \zeta) - P_a$.

mean over the full turbulence spectrum. Since the fluctuations of density can be taken as small with respect to their mean value, β and θ can be related by the linearised equation of state

$$\beta = g\beta_T\theta \tag{4.119}$$

The mean flow equations are obtained by substituting (4.118) into (4.115-4.117) and taking the average. This gives

$$\frac{\partial\overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial\overline{U_i}}{\partial x_j} + \epsilon_{ijk}f_j\overline{U_k} = -\frac{1}{\rho_0}\frac{\partial\overline{P}}{\partial x_i} + \delta_{i3}\overline{b} + \nu\sum_j\frac{\partial^2\overline{U_i}}{\partial x_j^2} - \frac{\partial}{\partial x_j}\overline{u_iu_j} \tag{4.120}$$

$$\frac{\partial\overline{U_i}}{\partial x_i} = 0 \tag{4.121}$$

$$\frac{\partial\overline{T}}{\partial t} + \overline{U_i}\frac{\partial\overline{T}}{\partial x_i} = \mathcal{P}(\overline{T}) + k_T\sum_i\frac{\partial^2\overline{T}}{\partial x_i^2} - \frac{\partial}{\partial x_i}\overline{u_i\theta} \tag{4.122}$$

since $\overline{\mathcal{P}(T)} = \mathcal{P}(\overline{T})$ in view of (4.82). Equations (4.120)–(4.122) are the same as (4.115)–(4.117) except for the last terms in the momentum and temperature equations which represent the exchange of momentum and heat between the mean and turbulent flows. The two terms appear as a divergence of fluxes of momentum $\overline{u_iu_j}$ and temperature $\overline{u_i\theta}$.

It remains to find suitable parameterisations for these turbulent fluxes. The oldest approach — dating back from the time of Boussinesq in 1877 — and also the most commonly used, is to model the momentum fluxes like the viscous stresses in laminar flows

$$-\overline{u_iu_j} = \nu_T\left(\frac{\partial\overline{U_i}}{\partial x_j} + \frac{\partial\overline{U_j}}{\partial x_i}\right) - \frac{2}{3}k\delta_{ij} \tag{4.123}$$

where

$$k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) \tag{4.124}$$

is the kinetic energy of turbulent motions and ν_T is called the eddy viscosity (diffusion) coefficient. In analogy with (4.123) the temperature fluxes are usually parameterised using the down-gradient diffusion hypothesis

$$-\overline{u_i\theta} = \lambda_T\frac{\partial\overline{T}}{\partial x_i} \tag{4.125}$$

Despite the similarity with laminar flows, there are fundamental differences between turbulent and laminar diffusion

- For fully developed turbulence, ν_T and λ_T are larger than their laminar counterparts by several orders of magnitude.

- The turbulent diffusion coefficients are of the same order of magnitude whereas the molecular coefficients for momentum, T and S are of the order of $(\nu, k_T, k_S) \sim (10^{-6}, 10^{-7}, 10^{-9})$ m^2/s.

- Turbulence is initiated by instabilities of the mean flow. This means in practice that ν_T and λ_T are not constant but depend on the mean flow properties generating and controling those instabilities, i.e. the current shear $\partial U_i/\partial x_j$ and the density gradient $\partial\rho/\partial x_i$.

The vertical diffusion terms in the model equations (4.44),(4.45) and (4.47)–(4.48) are then derived by evaluating the flux divergences using (4.123)–(4.125) and making the shallow water approximation. This condition is compatible with the assumption of hydrostatic balance and states that the horizontal (mean flow) scales are larger than the vertical one, or $\partial/\partial x, \partial/\partial y \ll \partial/\partial z$. This gives

$$\begin{aligned}
-\frac{\partial}{\partial x}\overline{u^2} - \frac{\partial}{\partial y}\overline{uv} - \frac{\partial}{\partial z}\overline{uw} \simeq -\frac{\partial}{\partial z}\overline{uw} \simeq \frac{\partial}{\partial z}\nu_T\frac{\partial\overline{U}}{\partial z} \\
-\frac{\partial}{\partial x}\overline{uv} - \frac{\partial}{\partial y}\overline{v^2} - \frac{\partial}{\partial z}\overline{vw} \simeq -\frac{\partial}{\partial z}\overline{vw} \simeq \frac{\partial}{\partial z}\nu_T\frac{\partial\overline{V}}{\partial z} \\
-\frac{\partial}{\partial x}\overline{u\theta} - \frac{\partial}{\partial y}\overline{v\theta} - \frac{\partial}{\partial z}\overline{w\theta} \simeq -\frac{\partial}{\partial z}\overline{w\theta} \simeq \frac{\partial}{\partial z}\lambda_T\frac{\partial\overline{T}}{\partial z}
\end{aligned} \tag{4.126}$$

since

$$-\,\overline{uw} \simeq \nu_T\frac{\partial U}{\partial z}\,, \quad -\overline{vw} \simeq \nu_T\frac{\partial V}{\partial z} \tag{4.127}$$

by application of the shallow water approximation to (4.123). A similar expression applies for the diffusion of salinity. It is clear that (u,v,T,S) in (4.43)–(4.45), (4.47)–(4.48) are now interpreted as statistical averages.

The turbulence, as stated in the beginning of this subsection, is now reduced to the implementation of suitable expressions for the turbulent diffusion coefficients. A large number of schemes, applied for hydraulic engineering and in the geophysical context, are available from the scientific literature. For detailed overviews, the reader is referred to the text books of Kantha & Clayson (2000a); Pope (2001) and the reviews by Rodi (1984); Burchard (2002).

The turbulence models, implemented in COHERENS, fall in three categories of increasing complexity

1. The simplest formulation is to set the diffusion coefficients to constant values
$$\nu_T = \nu_c\,, \quad \lambda_T = \lambda_c \tag{4.128}$$
where ν_c and λ_c are set be the user. Despite its simplicity, it is recommended not to use this form. Turbulence usuallly occurs in the surface and bottom boundary layers, which requires spatially varying coefficients.

2. Models using simplified empirical or semi-empirical (algebraic) relations, not derived from a turbulence closure theory.

3. Models obtained from the Reynolds averaged Navier-Stokes (RANS) equations. These models are physically more robust, but have a larger computational overhead. The basic assumptions (4.123) and (4.125) are then not assumed *a priori* but derived *a posteriori* from theory.

Implementation

The general type of turbulence scheme is selected with the switch iopt_vdif_coef:

0 : Vertical diffusion is set to zero.

1 : Constant diffusion coefficients.

2 : Algebraic schemes (Section 4.4.2).

3 : RANS models (Section 4.4.3).

4.4.2 Algebraic schemes

4.4.2.1 Richardson number dependent formulations

The Richardson number is defined by

$$Ri = \frac{N^2}{M^2} \tag{4.129}$$

where

$$N^2 = \frac{1}{h_3}\frac{\partial b}{\partial s} = \frac{g}{h_3}\Big(\beta_T\frac{\partial T}{\partial s} - \beta_S\frac{\partial S}{\partial s}\Big) \tag{4.130}$$

$$M^2 = \frac{1}{h_3^2}\Big[\Big(\frac{\partial u}{\partial s}\Big)^2 + \Big(\frac{\partial v}{\partial s}\Big)^2\Big] \tag{4.131}$$

are the squared buoyancy and shear frequencies. The first one measures the degree of stratification which is stable if $N^2 > 0$ ($Ri > 0$) and unstable if

$N^2 < 0$ ($Ri < 0$). The formulations below are based on theoretical and observational evidence that turbulence decreases in the first case and increases in the second one.

Pacanowski & Philander (1981) proposed the following formulation

$$\nu_T = \nu_{0p} f_p^{n_p}(Ri) + \nu_{bp} \tag{4.132}$$

$$\lambda_T = \nu_T f_p(Ri) + \lambda_{bp} \tag{4.133}$$

$$f_p(Ri) = \min\Big[(1+\alpha_p Ri)^{-1}, \nu_{max}^{1/n_p}\Big] \tag{4.134}$$

An upper limit has been imposed on f_p to prevent that turbulence becomes too large in the case of unstable stratification ($Ri < 0$).

The following default values are used[6]

$$\nu_{0p} = 10^{-2} \ , \ n_p = 2 \ , \ \alpha_p = 5 \ , \ \nu_{bp} = 10^{-4} \ , \ \lambda_{bp} = 10^{-5} \ , \ \nu_{max} = 3 \tag{4.135}$$

The upper bounds for ν_T and λ_T are then given by 0.03, respectively 0.052.

The scheme has been primarily developed for application in global ocean models (e.g. Semtner & Chervin, 1988). It has the advantage of being less sensitive to vertical resolution than the more advanced turbulence closures discussed in Section 4.4.3. In the absence of stratification the coefficients take uniform values which makes the scheme less reliable for the study of neutral tidal and wind-driven flows. Test simulations in the Rhine plume (Ruddick *et al.*, 1995) showed that the results are sensitive to a calibration of the model constants. Peters *et al.* (1988) derived a similar formulation using different values of the parameters calibrated from microstructure measurements in the Pacific Ocean.

The second scheme use the historical empirical relations proposed by Munk & Anderson (1948):

$$\nu_T = \nu_{0m} f_m(Ri) + \nu_b \tag{4.136}$$

$$\lambda_T = \nu_{0m} g_m(Ri) + \lambda_b \tag{4.137}$$

with

$$f_m(Ri) = \min\big[(1+\alpha_m Ri)^{-n_1}, \nu_{max}\big] \tag{4.138}$$

$$g_m(Ri) = \min\big[(1+\beta_m Ri)^{-n_2}, \lambda_{max}\big] \tag{4.139}$$

and ν_b and λ_b are uniform background mixing coefficients selected by the user. The following default parameter values taken

$$\nu_{0m} = 0.06 \ , \ \alpha_m = 10 \ , \ \beta_m = 3.33 \ , \ n_1 = 0.5 \ , \ n_2 = 1.5 \ , \ \nu_{max} = 3 \ , \ \lambda_{max} = 4 \tag{4.140}$$

[6]Note that ν_{0p}, ν_{bp}, λ_{bp} and ν_{0m} in (4.135) and (4.140) are expressed in the same unit as ν_T and λ_T, i.e. m^2/s, whereas ν_{max}, λ_{max} are dimensionless.

4.4.2.2 flow-dependent formulations

In shelf and coastal seas tides are a prominent source of turbulence. Observations in the Irish Sea (Bowden *et al.*, 1959) indicate that the eddy viscosity is proportional to the magnitude of the tidal current. A suitable parameterisation for tidal flow can then be written as

$$\nu_T = \big(\alpha(\xi_1,\xi_2,t)\Phi(\sigma) + \nu_w\big) f_m(Ri) + \nu_b \tag{4.141}$$

$$\lambda_T = \big(\alpha(\xi_1,\xi_2,t)\Phi(\sigma) + \nu_w\big) g_m(Ri) + \lambda_b \tag{4.142}$$

The flow field is represented by the depth-independent factor α. Explicit forms are described below. In the absence of stratification the vertical variation of turbulence is presented by a prescribed profile for $\Phi(\sigma)$ which takes account of the reduction of turbulence in the near bottom and surface layers. Following Davies (1990) the following piecewise linear profile is adopted

$$\begin{aligned}
\Phi(\sigma) &= \Big((1-r_1)\sigma/\delta_1 + r_1\Big)/D \qquad &&\text{for} \quad 0 \le \sigma \le \delta_1 \\
\Phi(\sigma) &= 1/D &&\text{for} \quad \delta_1 \le \sigma \le 1-\delta_2 \\
\Phi(\sigma)&\Big(r_2 - (r_2-1)(1-\sigma)/\delta_2\Big)/D &&\text{for} \quad 1-\delta_2 \le \sigma \le 1
\end{aligned} \tag{4.143}$$

where

$$D = 1 + \frac{1}{2}\delta_1(r_1-1) + \frac{1}{2}\delta_2(r_2-1) \tag{4.144}$$

is a normalisation factor such that the depth-integral of $\Phi(\sigma)$ equals 1. The parameters δ_1, δ_2 are the fractional depths of the bottom and surface layers, and r_1, r_2 the ratios of the bottom and surface values of Φ with respect to the interior value. Default values for the parameters are

$$\delta_1 = \delta_2 = 0\ ,\ r_1 = r_2 = 1 \tag{4.145}$$

giving a uniform vertical profile. More details about a proper selection of these parameters can be found in e.g. Davies (1993).

In analogy with the formulation used by Naimie *et al.* (1994) for simulating the circulation around Georges Bank the damping functions $f_m(Ri)$ and $g_m(Ri)$ take the form given by the Munk-Anderson expressions (4.138)–(4.139). Following Glorioso & Davies (1995) wind-induced turbulence is related to the surface friction velocity using the simple form

$$\nu_w = \lambda_\star u_{*s} \tag{4.146}$$

where $\lambda_\star$ is a constant tunable parameter and the surface friction velocity $u_{\star s}$ is given by

$$u_{\star s} = \tau_s^{1/2} = (\tau_{s1}^2 + \tau_{s2}^2)^{1/2} \tag{4.147}$$

The last terms on the right of (4.141)–(4.142) are the uniform background eddy viscosity ν_b and diffusivity λ_b.

The following three formulations for the flow factor α can be selected

$$\alpha = K_1(U^2+V^2)^{1/2} \quad (4.148)$$
$$\alpha = K_2(U^2+V^2)/(H^2\omega_1) \quad (4.149)$$
$$\alpha = K_1(U^2+V^2)^{1/2}\Delta_b/H \quad (4.150)$$

where Δ_b measures the thickness of the bottom boundary layer as a function of the bottom friction velocity u_{*b}:

$$\Delta_b = \min(C_\nu u_{*b}/\omega_1, H) \quad (4.151)$$

$$u_{*b} = \tau_b^{1/2} = (\tau_{b1}^2 + \tau_{b2}^2)^{1/2} \quad (4.152)$$

and ω_1 is a characteristic frequency. In shallow areas $\Delta_b = H$ so that (4.150) reduces to (4.148). The following default values are taken

$$K_1 = 2.5\times10^{-3}\ ,\ K_2 = 2\times10^{-5}\ ,\ C_\nu = 2.0\ ,\ \omega_1 = 10^{-4}\mathrm{s}^{-1} \quad (4.153)$$

The eddy viscosity parameterisation (4.141) without stratification has been used for the prediction of tidal currents and surface elevations in the Northwest European Continental Shelf (e.g. Davies, 1990; Davies *et al.*, 1997), the Irish and Celtic Seas (e.g. Davies & Jones, 1992; Davies, 1993) and the shelf edge off the West coast of Scotland (Proctor & Davies, 1996).

A parabolic eddy viscosity/diffusivity has been implemented for simplified test case studies

$$\begin{aligned}\nu_T &= \kappa f_m(Ri)u_{*b}H\sigma(1-\sigma)\\ \lambda_T &= \kappa g_m(Ri)u_{*b}H\sigma(1-\sigma)\end{aligned} \quad (4.154)$$

where f_m, g_m are the damping functions defined by (4.138)–(4.139).

Implementation

An algebraic scheme is taken if iopt_vdif_coef=2. The type of scheme is further selected by the switch iopt_turb_alg:

1 : Pacanowski-Philander

2 : Munk-Anderson

3 : Flow dependent scheme with α given by (4.148)

4 : Flow dependent scheme with α given by (4.149)

5 : Flow dependent scheme with α given by (4.150)

6 Parabolic profile (4.154)

4.4.3 RANS models

Contrary to the previous schemes, RANS models do not take the eddy viscosity-diffusivity concept as a prior assumption. The turbulent fluxes are derived from a series of transport equations (Section 4.4.3.1). These equations contain unknown second and third-order correlations which need to be parameterised (Section 4.4.3.2). Analytical solutions for geophysical flows and expresssions for the eddy viscosity and diffusivity coefficients are derived in Section 4.4.3.3. The solutions contain the turbulent energy k and its dissipation ε as unknown variables. Alternative formulations using a mixing length are given in Section 4.4.3.5. Different schemes are presented in Section 4.4.3.4. Alternative formulations using a mixing length are defined in Section 4.4.3.5. Background mixing schemes are discussed in Section 4.4.3.6.

4.4.3.1 general form of the RANS equations

Equations for the turbulent fluctuations u_i and β are obtained by substracting the mean equations (4.120)–(4.122) from the non-averaged equations (4.115)–(4.117) and making use of (4.119). One obtains

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_j}(U_iu_j+u_iU_j+u_iu_j)+\epsilon_{ijk}f_ju_k=-\frac{\partial\pi}{\partial x_i}+\beta\delta_{i3}+\frac{\partial}{\partial x_j}\overline{u_iu_j}+\nu\sum_j\frac{\partial^2u_i}{\partial x_j^2} \tag{4.155}$$

$$\frac{\partial u_i}{\partial x_i}=0 \tag{4.156}$$

$$\frac{\partial\beta}{\partial t}+\frac{\partial}{\partial x_i}(U_i\beta+g\beta_TTu_i+u_i\beta)=k_T\sum_i\frac{\partial^2\beta}{\partial x_i^2}+\frac{\partial\overline{u_i\beta}}{\partial x_i} \tag{4.157}$$

where the overbar has been omitted for convenience above mean quantities and the temperature equation is converted to an equation for the perturbed buoyancy β by multiplication with the factor $g\beta_T$.

Adding the i-component of (4.155) multiplied by u_j and the j-component multiplied by u_i and taking the average, the following system of equations is obtained for the Reynolds stresses $\overline{u_iu_j}$, making use of the zero divergence condition (4.156)

$$\begin{aligned}
&\frac{d}{dt}\overline{u_iu_j}+\frac{\partial}{\partial x_k}\overline{u_iu_ju_k}+f_k(\epsilon_{ikl}\overline{u_lu_j}+\epsilon_{jkl}\overline{u_lu_i})=\\
&\quad-\frac{\partial}{\partial x_i}\overline{u_j\pi}-\frac{\partial}{\partial x_j}\overline{u_i\pi}-\overline{u_iu_k}\frac{\partial U_j}{\partial x_k}-\overline{u_ju_k}\frac{\partial U_i}{\partial x_k}\\
&\quad+\delta_{i3}\overline{u_j\beta}+\delta_{j3}\overline{u_i\beta}+\overline{\pi\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}+\nu\frac{\partial^2}{\partial x_k^2}\overline{u_iu_j}-2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}
\end{aligned}$$

(4.158)

where use is made of the zero divergence condition (4.156) and the total derivative is defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k} \tag{4.159}$$

The transport equation for the buoyancy fluxes $\overline{u_i \beta}$ is derived by multiplying (4.155) by β, (4.157) by u_i adding and making the average

$$\begin{aligned} \frac{d}{dt}\overline{u_i\beta} + \frac{\partial}{\partial x_j}\overline{u_i u_j \beta} + \epsilon_{ijk} f_j \overline{u_k \beta} = \\ -\frac{\partial}{\partial x_i}\overline{\beta\pi} - \overline{u_i u_j}\frac{\partial b}{\partial x_j} - \overline{u_j\beta}\frac{\partial U_i}{\partial x_j} + \delta_{i3}\overline{\beta^2} + \overline{\pi\frac{\partial \beta}{\partial x_i}} \\ + \nu\frac{\partial}{\partial x_j}\overline{\beta\frac{\partial u_i}{\partial x_j}} + k_T\frac{\partial}{\partial x_j}\overline{u_i\frac{\partial \beta}{\partial x_j}} - (\nu + k_T)\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial \beta}{\partial x_j}} \end{aligned} \tag{4.160}$$

The β^2-equation is obtained by multiplying (4.157) with 2β and averaging

$$\frac{d}{dt}\overline{\beta^2} + \frac{\partial}{\partial x_i}\overline{u_i\beta^2} = -2\overline{u_i\beta}\frac{\partial b}{\partial x_i} + k_T\sum_i \frac{\partial^2\overline{\beta^2}}{\partial x_i^2} - 2k_T\sum_i \overline{\left(\frac{\partial \beta}{\partial x_i}\right)^2} \tag{4.161}$$

Equations (4.158)–(4.161) form a complete set of equations for all second-order correlations. An important equation is the one for turbulent kinetic energy k, defined by (4.124) and obtained from (4.158) by taking half its trace

$$\begin{aligned} \frac{dk}{dt} + \frac{\partial}{\partial x_j}\overline{u_i(\frac{1}{2}u_i u_j + \pi)} - \nu\frac{\partial^2 k}{\partial x_i^2} &= -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} + \delta_{i3}\overline{u_i\beta} - \nu\overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} \\ &= P + G - \varepsilon \end{aligned} \tag{4.162}$$

where summation is taken over all indices. The terms on the right have the following meaning:

- P is a source term and equals the energy withdrawn by the “energy containing eddies” at the largest spatial scales of the turbulence spectrum
- G is a buoyancy term which can shown to be (see below) a sink term for stable stratification ($Ri > 0$) and a source term for an unstable stratification ($Ri < 0$).
- ε represents the dissipation of turbulent energy which occurs at the smallest scales of turbulence.

One main properties of fully developed turbulence is that it adjusts rapidly to a state of equilibrium. This means that the terms on the right hand side are the dominant ones and $P + G \simeq \varepsilon$. A further advantage is that P and G don't contain unknown correlations.

4.4.3.2 parameterisation of the RANS equations

The following parameterisations are adopted for the unknown correlations in (4.158), (4.160) and (4.161).

- All Coriolis terms are set to zero. The main reason is that rotation introduces a large level of complexity. The simplification can be considered as reasonable when the Coriolis period is much larger than the decay time of turbulence or $f\varepsilon/k \ll 1$. For an account of Coriolis effects see Galperin *et al.* (1989); Kantha *et al.* (1989).

- Pressure-strain correlation

$$\begin{aligned}\overline{\pi\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)} &= -c_1\frac{\varepsilon}{k}(\overline{u_iu_j}-\frac{2}{3}\delta_{ij}k) - c_{21}(P_{ij}-\frac{2}{3}\delta_{ij}P)\\ &\quad -c_{22}k\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right) - c_{23}(D_{ij}-\frac{2}{3}\delta_{ij}P)\\ &\quad -c_3(G_{ij}-\frac{2}{3}\delta_{ij}G) \qquad (4.163)\end{aligned}$$

  The first term represents the Rotta (1951) hypothesis of return to isotropy. The tensors in the remaining terms are defined by

$$P_{ij} = -\overline{u_iu_k}\frac{\partial U_j}{\partial x_k} - \overline{u_ju_k}\frac{\partial U_i}{\partial x_k} \qquad (4.164)$$

$$D_{ij} = -\overline{u_iu_k}\frac{\partial U_k}{\partial x_j} - \overline{u_ju_k}\frac{\partial U_k}{\partial x_i} \qquad (4.165)$$

$$G_{ij} = \delta_{i3}\overline{u_j\beta} + \delta_{j3}\overline{u_i\beta} \qquad (4.166)$$

$$P = \frac{1}{2}P_{ii} = -\overline{u_iu_k}\frac{\partial U_i}{\partial x_k} \qquad (4.167)$$

$$G = \frac{1}{2}G_{ii} = \delta_{i3}\overline{u_i\beta} = \overline{w\beta} \qquad (4.168)$$

- Pressure-buoyancy gradient correlation

$$\overline{\pi\frac{\partial\beta}{\partial x_i}} = -c_{1\beta}\frac{\varepsilon}{k}\overline{u_i\beta} + c_{21\beta}\overline{u_k\beta}\frac{\partial U_i}{\partial x_k} - c_{22\beta}\overline{u_k\beta}\frac{\partial U_k}{\partial x_i} - c_{3\beta}\delta_{i3}\overline{\beta^2} \qquad (4.169)$$

Table 4.2: Values of the turbulence constants in the RANS equations according to the different models implemented in COHERENS.

	c_1	c_{21}	c_{22}	c_{23}	c_3	$c_{1\beta}$	$c_{21\beta}$	$c_{22\beta}$	$c_{3\beta}$	R_β
MY82	3.01	0	-0.16	0	0	3.74	0	0	0	0.61
KC94	3.01	0	-0.16	0	0	3.74	0.7	0	0.2	0.61
BB95	1.8	0.6	0	0	0.6	3.0	0.33	0	0.333	0.8
HR82	2.2	0.55	0	0	0.55	3.0	0.5	0	0.5	0.8
CA01	2.49	0.777	0.256	0.207	0.402	5.95	0.8	0.2	0.333	0.72
CA02	2.1	0.803	0.257	0.183	0.576	5.6	0.8	0.2	0.333	0.477

- Dissipation terms

$$2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} = \frac{2}{3}\delta_{ij}\varepsilon \qquad (4.170)$$

$$2k_T\overline{\left(\frac{\partial\beta}{\partial x_i}\right)^2} = \chi = \frac{\varepsilon}{k}\frac{\overline{\beta^2}}{R_\beta} \qquad (4.171)$$

$$\nu\frac{\partial}{\partial x_j}\overline{\beta\frac{\partial u_i}{\partial x_j}} = k_T\frac{\partial}{\partial x_j}\overline{u_i\frac{\partial\beta}{\partial x_j}} = (\nu+k_T)\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial\beta}{\partial x_j}} = 0 \qquad (4.172)$$

Since the laminar diffusion scales are much smaller than the scales of the largest eddies, the laminar terms can be neglected

$$\nu\frac{\partial^2}{\partial x_k^2}\overline{u_iu_j} = 0\,, \quad k_T\frac{\partial^2\overline{\beta^2}}{\partial x_i^2} = 0 \qquad (4.173)$$

Equation(4.170) states that turbulence energy is dissipated isotropically. Equation(4.171) assumes that the dissipation time scale of turbulence kinetic enery k/ε is proportional to the one for the buoyancy variance $\overline{\beta^2}/\chi$. The ratio is given by the parameter R_β.

- Pressure transport is neglected

$$\frac{\partial}{\partial x_i}\overline{u_j\pi} = 0\,, \quad \frac{\partial}{\partial x_i}\overline{\beta\pi} = 0 \qquad (4.174)$$

The expressions for the pressure-strain and pressure-buoyancy gradient are compiled from different sources and presented in their most general form. The parameterisations contain 10 parameters c_1, c_{21}, c_{22}, c_{23}, c_3, $c_{1\beta}$, $c_{21\beta}$, $c_{22\beta}$, $c_{3\beta}$, R_β. The values used in COHERENS are taken from different sources: Mellor & Yamada (1982); Kantha & Clayson (1994); Burchard & Baumert

(1995); Hossain & Rodi (1982) and two schemes taken from Canuto *et al.* (2001). The six models are further denoted by MY82, KC94, BB95, HR82, CA01 and CB01. Values of the parameters are listed in Table 4.2. For readers, familiar with the A_i, B_i, C_i parameters used in MY82 and KC94, the following conversion rules apply

$$
\begin{aligned}
c_1 &= \frac{B_1}{6A_1}\,,\; c_{21} = 0\,,\; c_{22} = -2C_1\,,\; c_{23} = 0\,,\; c_3 = 0 \\
c_{1\beta} &= \frac{B_1}{6A_2}\,,\; c_{21\beta} = C_2\,,\; c_{22\beta} = 0\,,\; c_{3\beta} = C_3\,,\; R_\beta = \frac{B_2}{B_1}
\end{aligned}
\tag{4.175}
$$

Different schemes are available for the modelisation of the third-order correlations, total derivative (i.e. time derivative and advective terms) in (4.158)–(4.162). They are based on the classification scheme introduced by Mellor & Yamada (1974, 1982) and Galperin *et al.* (1988).

1. Non-equilibrium or "level 3" method. The left hand sides of (4.160) and (4.161) are set to zero while

$$
\frac{d}{dt}\overline{u_i u_j} + \frac{\partial}{\partial x_k}\overline{u_i u_j u_k} = \frac{2}{3}\delta_{ij}(P + G - \varepsilon) \tag{4.176}
$$

   in (4.158) and

$$
\frac{1}{2}\frac{\partial}{\partial x_j}\overline{u_i^2 u_j} = -c_{sk}\frac{\partial}{\partial x_j}\Big(\frac{k}{\varepsilon}\overline{u_j u_k}\frac{\partial k}{\partial x_k}\Big) \tag{4.177}
$$

   in (4.162). The expression in the last equation was proposed by Daly & Harlow (1970) and differs somewhat from the one given in the Mellor-Yamada papers. This will be further discussed below.

2. Quasi-equilibrium or "level 2.5" method after Galperin *et al.* (1988). This is the same as previous except that $P+G$ is set to ε in all equations except in the one for turbulent energy. This mean that the left hand side of (4.158) becomes zero as well.

3. Equilibrium or "level 2" method. A full equilibrium, i.e. $P + G = \varepsilon$ in all equations including the k-equation.

A major simplification is additionally achieved by making the boundary layer approximation. This means that horizontal derivatives of mean quantities and the mean vertical current are all set to zero.

4.4.3.3 stability functions

Substituting the parameterisations and approximations, presented in the previous subsection into the equations for the second-order correlations, the problem reduces to solving a system of linear equations. Solutions are presented below for each of the three equilibrium levels.

Non-equilibrium method

The following system of linear equations is obtained where a vertical derivative is denoted by a subscript z and $b_z = N^2$ by its definition (4.130)

$$
\begin{aligned}
\overline{u^2} &= \frac{2}{3}\Big[k - \frac{k}{\varepsilon c_1}\Big((2-2c_{21}+c_{23})\overline{uw}U_z - (1-c_{21}-c_{23})\overline{vw}V_z + (1-c_3)\overline{w\beta}\Big)\Big] \\
\overline{v^2} &= \frac{2}{3}\Big[k + \frac{k}{\varepsilon c_1}\Big((1-c_{21}-c_{23})\overline{uw}U_z - (2-2c_{21}+c_{23})\overline{vw}V_z - (1-c_3)\overline{w\beta}\Big)\Big] \\
\overline{w^2} &= \frac{2}{3}\Big[k + \frac{k}{\varepsilon c_1}\Big((1-c_{21}+2c_{23})(\overline{uw}U_z + \overline{vw}V_z) + 2(1-c_3)\overline{w\beta}\Big)\Big] \\
\overline{uv} &= -\frac{k}{\varepsilon c_1}(1-c_{21})(\overline{uw}V_z + \overline{vw}U_z) \\
\overline{uw} &= -\frac{k}{\varepsilon c_1}\Big[(1-c_{21})\overline{w^2}U_z + c_{22}kU_z - c_{23}(\overline{u^2}U_z + \overline{uv}V_z) - (1-c_3)\overline{u\beta}\Big] \\
\overline{vw} &= -\frac{k}{\varepsilon c_1}\Big[(1-c_{21})\overline{w^2}V_z + c_{22}kV_z - c_{23}(\overline{v^2}V_z + \overline{uv}U_z) - (1-c_3)\overline{v\beta}\Big] \\
\overline{u\beta} &= -\frac{k}{\varepsilon c_{1\beta}}\Big[\overline{uw}N^2 + (1-c_{21\beta})\overline{w\beta}U_z\Big] \\
\overline{v\beta} &= -\frac{k}{\varepsilon c_{1\beta}}\Big[\overline{vw}N^2 + (1-c_{21\beta})\overline{w\beta}V_z\Big] \\
\overline{w\beta} &= -\frac{k}{\varepsilon c_{1\beta}}\Big[\overline{w^2}N^2 + c_{22\beta}(\overline{u\beta}U_z + \overline{v\beta}V_z) - (1-c_{3\beta})\overline{\beta^2}\Big] \\
\overline{\beta^2} &= -2\frac{k}{\varepsilon}R_\beta N^2\overline{w\beta}
\end{aligned}
\tag{4.178}
$$

The solutions for the vertical fluxes of momentum and buoyancy can be written as

$$
-\overline{uw} = S_u\frac{k^2}{\varepsilon}\frac{\partial U}{\partial z}\,, \quad -\overline{vw} = S_u\frac{k^2}{\varepsilon}\frac{\partial V}{\partial z}\,, \quad -\overline{w\beta} = S_b\frac{k^2}{\varepsilon}N^2 \tag{4.179}
$$

Comparing with (4.127) and (4.125) the eddy viscosity and diffusivity are given by

$$
\nu_T = S_u\frac{k^2}{\varepsilon}\,, \quad \lambda_T = S_b\frac{k^2}{\varepsilon} \tag{4.180}
$$

Note that these expressions are obtained from the RANS theory and not postulated *a priori*.

The coefficients S_u and S_b are the so-called stability functions and can be expressed as function of the stability parameters

$$
\alpha_M = \frac{k^2}{\varepsilon^2}M^2\,, \quad \alpha_N = \frac{k^2}{\varepsilon^2}N^2 \tag{4.181}
$$

The two parameters are the squares of the ratio of respectively the shear and buoyancy frequency with respect to the turbulence frequency and represent the influence of shear and density gradients on the turbulent fluxes. The solutions for S_u and S_b can be written as

$$\begin{aligned} S_u &= (C_{a1} + C_{a2}\alpha_M + C_{a3}\alpha_N)/D \\ S_b &= (C_{a4} + C_{a5}\alpha_M + C_{a6}\alpha_N)/D \\ D &= 1 + C_{a7}\alpha_M + C_{a8}\alpha_N + C_{a9}\alpha_M^2 + C_{a10}\alpha_M\alpha_N + C_{a11}\alpha_N^2 \end{aligned} \quad (4.182)$$

Explicit expressions for the coefficients C_{ai} as function of the RANS parameters are given in Appendix B.

When applying the boundary layer approximation, the diffusion term (4.177) in the k-equation can be written as

$$c_{sk}\frac{\partial}{\partial x_j}\Big(\frac{k}{\varepsilon}\overline{u_j u_k}\frac{\partial k}{\partial x_k}\Big) \simeq c_{sk}\frac{\partial}{\partial z}\Big(\frac{k}{\varepsilon}\overline{w^2}\frac{\partial k}{\partial z}\Big) = \frac{\partial}{\partial z}\Big(\nu_k\frac{\partial k}{\partial z}\Big) \quad (4.183)$$

where the diffusion coefficient is given by

$$\nu_k = c_{sk}\frac{k^2}{\varepsilon}\frac{\overline{w^2}}{k} = S_k\frac{k^2}{\varepsilon} \quad (4.184)$$

The stability coefficient for turbulent energy diffusion is given by

$$S_k = c_{sk}\frac{\overline{w^2}}{k} = \frac{2}{3}c_{sk}\Big[1 - \frac{1}{c_1}\Big((1 - c_{21} + 2c_{23})\alpha_M S_u + 2(1 - c_3)\alpha_N S_b\Big)\Big] \quad (4.185)$$

Quasi-equilibrium method

The equations for the second order correlations are the same as in (4.178) except for the components of turbulent energy

$$\begin{aligned} \overline{u^2} &= \frac{2}{3}k\Big(1 - \frac{1 - c_{21} - c_{23}}{c_1}\Big) - \frac{2k}{\varepsilon c_1}\Big[(1 - c_{21})\overline{uw}U_z + \frac{1}{3}(c_{21} + c_{23} - c_3)\overline{w\beta}\Big] \\ \overline{v^2} &= \frac{2}{3}k\Big(1 - \frac{1 - c_{21} - c_{23}}{c_1}\Big) - \frac{2k}{\varepsilon c_1}\Big[(1 - c_{21})\overline{vw}V_z + \frac{1}{3}(c_{21} + c_{23} - c_3)\overline{w\beta}\Big] \\ \overline{w^2} &= \frac{2}{3}k\Big(1 - \frac{1 - c_{21} + 2c_{23}}{c_1}\Big) + \frac{2k}{3\varepsilon c_1}\Big((3 - c_{21} + 2_{23} - 2c_3)\overline{w\beta}\Big) \end{aligned} \quad (4.186)$$

The solutions for the vertical fluxes can be written in the form (4.179)–(4.181), obtained with the non-equilibrium method. Difference is that the stability functions now only depend on α_N. Two cases can be distinguished

1. Case $c_{22\beta} = 0$.

$$\begin{aligned} S_u &= \frac{C_{b1} + C_{b2}\alpha_N}{(1 + C_{b3}\alpha_N)(1 + C_{b4}\alpha_N)} \\ S_b &= \frac{C_{b5}}{1 + C_{b3}\alpha_N} \end{aligned} \tag{4.187}$$

2. Case $c_{22\beta} \neq 0$.

$$S_u = \frac{C_{c1} + C_{c2}\alpha_N S_b}{1 + C_{c3}\alpha_N} \tag{4.188}$$

S_b is obtained as solution of the quadratic equation

$$(C_{c4} + C_{c5}\alpha_N)\alpha_N S_b^2 + (C_{c6} + C_{c7}\alpha_N)S_b + C_{c8} = 0 \tag{4.189}$$

giving

$$\begin{aligned} S_b &= -\frac{(C_{c6} + C_{c7}\alpha_N) + sD^{1/2}}{2(C_{c4} + C_{c5}\alpha_N)\alpha_N} \\ D &= (C_{c6} + C_{c7}\alpha_N)^2 - 4C_{c8}(C_{c4} + C_{c5}\alpha_N)\alpha_N \end{aligned} \tag{4.190}$$

where s is the sign of $C_{c6} + C_{c7}\alpha_N$.

The expression for the turbulent energy stability coefficient now becomes

$$S_k = c_{sk}\frac{\overline{w^2}}{k} = \frac{2c_{sk}}{3c_1}\Big(c_1 - 1 + c_{21} - 2c_{23} - (3 - c_{21} + 2c_{23} - 2c_3)\alpha_N S_b\Big) \tag{4.191}$$

Equilibrium method

In this case equilibrium between production and dissipation of turbulent energy is assumed in all second moment equation and in the equation of turbulent energy. This means that

$$P + G = \varepsilon \quad \text{or in dimensionless form} \quad S_u\alpha_M - S_b\alpha_N = 1 \tag{4.192}$$

since

$$\frac{P+G}{\varepsilon} = -\frac{\overline{uw}U_z + \overline{vw}V_z}{\varepsilon} + \frac{\overline{w\beta}}{\varepsilon} = S_u\frac{k^2}{\varepsilon^2}M^2 - S_b\frac{k^2}{\varepsilon^2}N^2 = S_u\alpha_M - S_b\alpha_N \tag{4.193}$$

Using (4.192) and the expressions for S_u and S_b, the following relation between the stability parameters can be derived

$$1 + C_{d1}\alpha_M + C_{d2}\alpha_N + C_{d3}\alpha_M^2 + C_{d4}\alpha_M\alpha_N + C_{d5}\alpha_N^2 = 0 \tag{4.194}$$

which can be rewritten as a quadratic equation for the squared inverse time scale $\kappa^2 = \varepsilon^2/k^2$

$$\kappa^4 + (C_{d1}M^2 + C_{d2}N^2)\kappa^2 + C_{d3}M^4 + C_{d4}M^2N^2 + C_{d5}N^4 = 0 \qquad (4.195)$$

If the limit $\kappa = 0$ is taken, then $\alpha_M = \alpha_N = \infty$ so that $S_u = S_b = 0$. From (4.194) one therefore deduces that turbulence ceases when the Richardson number exceeds a critical value Ri_c obtained by setting $\kappa^2 = 0$. This gives

$$\begin{aligned} Ri_c &= -\frac{C_{d4}}{C_{d5}} \quad \text{if} \quad c_{22\beta} = 0 \\ Ri_c &= \frac{\sqrt{C_{d4}^2 - 4C_{d3}C_{d5}} - C_{d4}}{2C_{d5}} \quad \text{if} \quad c_{22\beta} \neq 0 \end{aligned} \qquad (4.196)$$

From Table 4.3 it is seen that Ri_c strongly depends on the type of model.

Since $\alpha_M = \alpha_N/Ri$ equation (4.194) can be rewritten as

$$(1 + C_{d2}\alpha_N + C_{d5}\alpha_N^2)Ri^2 + \alpha_N(C_{d1} + C_{d4}\alpha_N)Ri + C_{d3}\alpha_N^2 = 0 \qquad (4.197)$$

Solving (4.197) for Ri as a function of α_N and substituting into the expressions (4.187)–(4.190), the stability functions can be expressed as function of Ri only. The dependence of S_u and S_b on Ri is shown in Figure 4.7 for the six RANS models. Also shown is the case of stable stratification with limiting conditions which is further discussed in Section 4.4.3.6.

Alternative methods

Besides the expressions given above, the COHERENS program allows to use simpler formulations for the stability coefficients. In the first one the quasi-equilibrium expressions (4.187)–(4.190) are reset to their constant neutral values obtained in the absence of stratification ($\alpha_N = 0$)

$$S_u = S_{u0}\,, \quad S_b = S_{b0} \qquad (4.198)$$

Values of S_{u0} and S_{b0} are given in Table 4.3[7]. In the second form, S_u and S_b are set to their neutral values, multiplied by the Munk-Anderson damping/amplification factors, defined by (4.138) and (4.139) or

$$S_u = S_{u0}f_m(Ri)\,, \quad S_b = S_{b0}g_m(Ri) \qquad (4.199)$$

The latter scheme resembles the one adopted in the earlier standard version of the $k - \varepsilon$ model (Rodi, 1984).

[7]Node that S_{u0} is the same as the parameter c_μ used in the standard $k - \varepsilon$ theory (Rodi, 1984).

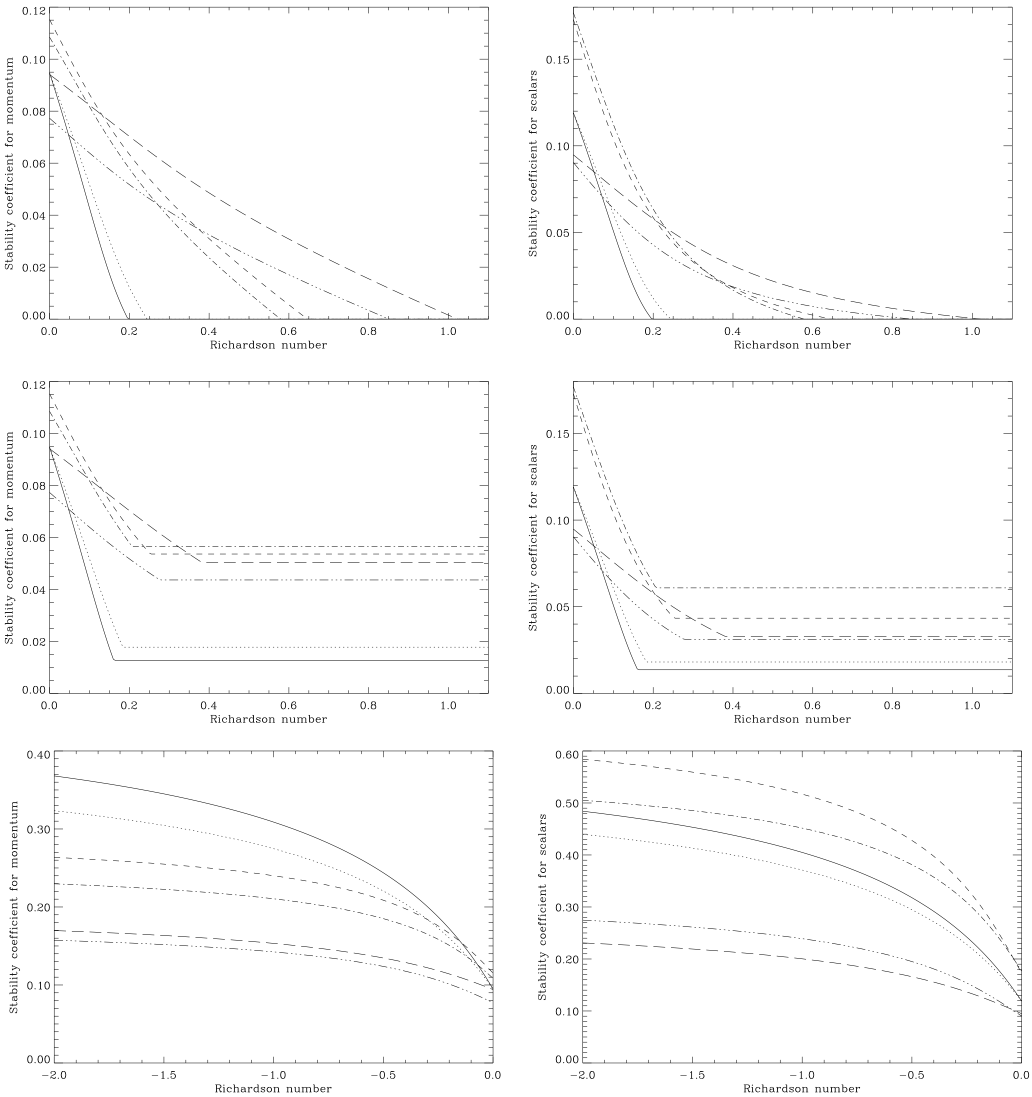

Figure 4.7: Stability coefficients S_u (left column), S_b (right column) as a function of the Richardson number using the equilibrium method for RANS model MY82 (solid), KC94 (dots), BB95 (dashes), HR82 (dash-dots), CA01 (dash and 3 dots), CA02 (long dashes): stable case without limiting condition (upper row), stable case with limiting condition (middle row), unstable case (bottom row).

Table 4.3: Values of critical parameters according to different RANS models.

	S_{u0}	S_{b0}	Ri_c	$Ri_c^\star$	α_{max}	ν_{0l}	λ_{0l}
MY82	0.095	0.119	0.197	0.161	15.35	0.050	0.053
KC94	0.095	0.119	0.244	0.183	12.69	0.063	0.065
BB95	0.115	0.173	0.647	0.252	5.92	0.130	0.106
HR82	0.108	0.177	0.579	0.207	4.72	0.123	0.132
CA01	0.077	0.090	0.851	0.281	7.91	0.123	0.088
CA02	0.094	0.095	1.023	0.388	10.07	0.160	0.104

It is remarked that these schemes are physically less robust, but have been implemented in the code for historical reasons or to perform sensitivity experiments related to specific details of the turbulence schemes.

Besides the general expressions (4.185) or (4.191) for the diffusion of turbulent energy, the following simpler alternative options for the stability coefficient S_k are implemented

$$S_k = S_{k0} \tag{4.200}$$

and

$$S_k = S_u/\sigma_k \tag{4.201}$$

where S_{k0} and σ_k are model constants. The first form was introduced in $k-l$ model of Mellor & Yamada (1982), the second in the “standard” $k-\varepsilon$ model (Rodi, 1984), further discussed below. In the former case S_{k0} is related to the Mellor-Yamada parameter S_q by

$$S_{k0} = 2^{1/2}\epsilon_0 S_q \tag{4.202}$$

with $\epsilon_0 = S_{u0}^{3/4}$ (see equation (4.203) below).

4.4.3.4 solution methods

The theories presented in Section 4.4.3.3 define the eddy coefficients as function of two turbulent parameters: turbulence energy k and its dissipation rate ε. Such theories, primarily used in hydraulic modelling, are called $k-\varepsilon$ models. The former is determined by solving either the equation for turbulent energy or by the equilibrium relation (4.195). It remains to determine ε appearing in the expressions for the eddy coefficients, stability parameters, k-equation and the equilibrium relation (4.195). Different methods, using either a prescribed analytical expression or a parameterised transport equation, are given below.

Other theories, like the one advocated by Mellor & Yamada (1974, 1982), prefer to use the mixing lenth l, representative of the spatial scales of the largest (energy-containing) eddies, instead. Such theories are called $k-l$ (or $q^2 - q^2 l$ with $q^2 = 2k$ in the Mellor-Yamada terminology) models. The $k-\varepsilon$ and $k-l$ models are separately discussed below.

$k-\varepsilon$ *models*

The schemes are divided into three categories depending on the number of transport equations which need to be solved.

1. Zero-equation model. The dissipation rate is determined through the relation
$$\varepsilon = \epsilon_0 \frac{k^{3/2}}{l} \tag{4.203}$$
where l is the mixing length and $\epsilon = S_{u0}^{3/4}$ where S_{u0} is the neutral value of the momentum stability coefficient S_u. The mixing length is determined from one of the available analytical expressions, given below. Turbulence energy is calculated from the equilibrium relation (4.195). The method has the advantage that no transport equation need to be solved in time. The problem is, however, that it may produce numerical instabilities in the time-integration of the model equations. This is illustrated in Section 24.1 for the test case ***pycno***.

2. One-equation model. Turbulence energy is obtained from the k-equation. Inserting the parameterisations of the previous subsection into the source and sinks terms P and G and the diffusion term, this equation, written in transformed coordinates and in its most general form, becomes
$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 k) + \mathcal{A}_{h1}(k) + \mathcal{A}_{h2}(k) + \mathcal{A}_v(k) = \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\nu_k}{h_3}\frac{\partial k}{\partial s}\Big) + \nu_T M^2 - \lambda_T N^2 - \varepsilon + \mathcal{D}_{sh1}(k) + \mathcal{D}_{sh2}(k) \tag{4.204}$$
with ε given by (4.203) and the advection and diffusion operators defined by (4.64)–(4.66), (4.77)–(4.78). The last two terms represent horizontal diffusion of k. It is noted that advection and horizontal diffusion are usually neglected. This is achieved in the model by the settings of two switches (see below).

3. Two-equation model. Besides the k-equation (4.204), a second transport equation is solved for ε. In transformed cordinates, this equation, including extra horizontal diffusion terms which are usually neglected, is given by (e.g. Rodi, 1984)
$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3\varepsilon) + \mathcal{A}_{h1}(\varepsilon) + \mathcal{A}_{h2}(\varepsilon) + \mathcal{A}_v(\varepsilon) = \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\nu_k}{h_3\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial s}\Big)$$

$$+ c_{1\varepsilon}\frac{\varepsilon}{k}\Big(\nu_T M^2 - c_{3\varepsilon}\lambda_T N^2\Big) - c_{2\varepsilon}\frac{\varepsilon^2}{k} + \mathcal{D}_{sh1}(\varepsilon) + \mathcal{D}_{sh2}(\varepsilon) \tag{4.205}$$

It should be remarked that, contrary to the k-equation, the ε-equation is derived from a general equation after many parameterisations. The parameters appearing in the (4.205) must satisfy the following constraint, derived from wall layer theory

$$c_{1\varepsilon} - c_{2\varepsilon} = -\frac{\kappa^2 S_{k0}}{\epsilon_0^2\sigma_\varepsilon} \tag{4.206}$$

where S_{u0} and S_{k0} are the neutral values of S_u and S_k.

$k - l$ models

In $k - l$ theory all equations are written explicitly as function of k and l. This is achieved by substituting l for ε through the relation (4.203). The expressions for the eddy viscosity and diffusion coefficients then take the form

$$\nu_T = S_m k^{1/2} l\,, \quad \lambda_T = S_h k^{1/2} l\,, \quad \nu_k = \tilde{S}_k k^{1/2} l \tag{4.207}$$

where the stability functions $(S_m, S_h, \tilde{S}_k) = (S_u, S_b, S_k)/\epsilon_0$ are function of the stability parameters $(G_m, G_h) = (M^2, N^2)\, l^2/k = \epsilon_0^2(\alpha_M, \alpha_N)$. Equations (4.182), (4.187), (4.188)–(4.190) and (4.194) remain valid with (α_M, α_N) replaced by (G_m, G_h), (S_u, S_b) by (S_m, S_h), κ by $\tilde{\kappa}$ and $(C_{ai}, C_{bi}, C_{ci}, C_{di})$ replaced by $(\tilde{C}_{ai}, \tilde{C}_{bi}, \tilde{C}_{ci}, \tilde{C}_{di})$ using

$$(\tilde{C}_{a1}, \tilde{C}_{a4}) = \epsilon_0(C_{a1}, C_{a4})\,, \quad (\tilde{C}_{a2}, \tilde{C}_{a3}, \tilde{C}_{a5}, \tilde{C}_{a6}) = \frac{1}{\epsilon_0}(C_{a2}, C_{a3}, C_{a5}, C_{a6})$$

$$(\tilde{C}_{a7}, \tilde{C}_{a8}) = \frac{1}{\epsilon_0^2}(C_{a7}, C_{a8})\,, \quad (\tilde{C}_{a9}, \tilde{C}_{a10}, \tilde{C}_{a11}) = \frac{1}{\epsilon_0^4}(C_{a9}, C_{a10}, C_{a11})$$

$$(\tilde{C}_{b1}, \tilde{C}_{b5}) = \epsilon_0(C_{b1}, C_{b5})\,, \quad \tilde{C}_{b2} = \frac{C_{b2}}{\epsilon_0}\,, \quad (\tilde{C}_{b3}, \tilde{C}_{b4}) = \frac{1}{\epsilon_0^2}(C_{b3}, C_{b4})$$

$$(\tilde{C}_{c1}, \tilde{C}_{c8}) = \epsilon_0(C_{c1}, C_{c8})\,, \quad \tilde{C}_{c6} = C_{c6}$$

$$(\tilde{C}_{c2}, \tilde{C}_{c3}, \tilde{C}_{c7}) = \frac{1}{\epsilon_0^2}(C_{c2}, C_{c3}, C_{c7})\,, \quad \tilde{C}_{c4} = \frac{1}{\epsilon_0^3}C_{c4}\,, \quad \tilde{C}_{c5} = \frac{1}{\epsilon_0^5}C_{c5}$$

$$(\tilde{C}_{d1}, \tilde{C}_{d2}) = \frac{1}{\epsilon_0^2}(C_{d1}, C_{d2})\,, \quad (\tilde{C}_{d3}, \tilde{C}_{d4}, \tilde{C}_{d5}) = \frac{1}{\epsilon_0^4}(C_{d3}, C_{d4}, C_{d5}) \tag{4.208}$$

In case of a two-equation model, the ϵ-equation is replaced by an equation for the quantity kl obtained from the Mellor-Yamada q^2l-equation by setting $q^2 = 2k$:

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 kl) + \mathcal{A}_{h1}(kl) + \mathcal{A}_{h2}(kl) + \mathcal{A}_v(kl) = \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\nu_k}{h_3\sigma_{kl}}\frac{\partial}{\partial s}(kl)\Big) + \frac{1}{2}l\Big(E_1\nu_T M^2 - E_3\lambda_T N^2\Big) - \frac{1}{2}\epsilon_0 k^{3/2}\tilde{W} + \mathcal{D}_{sh1}(kl) + \mathcal{D}_{sh2}(kl) \tag{4.209}$$

with the wall proximity function $\tilde{W}$ defined by

$$\tilde{W} = 1 + E_2\Big[\frac{l}{\kappa}\Big(\frac{1}{H(1-\sigma)+z_{0s}} + \frac{1}{\sigma H + z_{0b}}\Big)\Big]^2 \tag{4.210}$$

where z_{0s} and z_{0b} are roughness lengths at the surface, respectively the bottom (see below). In analogy with the ε-equation the parameters in (4.209)–(4.210) satisfy the constraint

$$E_2 - E_1 + 1 = \frac{2\kappa^2 S_{k0}}{\epsilon_0^2\sigma_{kl}} \tag{4.211}$$

4.4.3.5 mixing length formulations

A mixing length formulation is needed in the case of a zero- or one-equation model. Four formulations are implemented in the program. The basic requirement in each formulation is that l reduces to the following forms near, respectively, the bottom and the surface

$$l \simeq l_1 = \kappa(\sigma H + z_{0b})\ ,\ l \simeq l_2 = \kappa(H - \sigma H + z_{0s}) \tag{4.212}$$

where z_{0s} and z_{0b} are the surface and bottom roughness lengths.

The first and simplest expression is the parabolic law

$$\frac{1}{l} = \frac{1}{l_1} + \frac{1}{l_2} \tag{4.213}$$

having a maximum at $\sigma \simeq 0.5$.

The second is the "quasi-parabolic" law given by

$$\frac{1}{l} = \frac{1}{l_1}\Big(\frac{l_1 + l_2}{l_2}\Big)^{1/2} \tag{4.214}$$

which differs from the first one in that l has a maximum at $\sigma \simeq 2/3$ closer to the surface.

The third, recommended by Xing & Davies (1996) has the same form as (4.213) but with l_1 replaced by

$$l_1 = \kappa(\sigma H e^{-\beta_1\sigma} + z_{0b}) \tag{4.215}$$

allowing for a larger reduction of the mixing length in the lower parts of the water column.

The fourth formulation, initially proposed by Blackadar (1962), has the form

$$\frac{1}{l} = \frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_a} \tag{4.216}$$

so that $l \to l_a$ far from the boundaries. Mellor & Yamada (1974) defined l_a as the ratio of the first to the zeroth moment of the vertical profile of the turbulent velocity scale $k^{1/2}$. Hence

$$l_a = \alpha_1 \int_0^1 (1-\sigma) k^{1/2} H \, d\sigma \, / \int_0^1 k^{1/2} H \, d\sigma \tag{4.217}$$

4.4.3.6 background mixing

The theory, presented above, is valid for the case of a nearly-isotropic, fully developed turbulence under homeogeneous or weakly stratified conditions, but becomes invalid for strongly stratified flows. The reasons are as follows:

- It is known from theory and laboratory experiments that vertical turbulent excursions are impeded by stable stratification. The consequence is that turbulence becomes anisotropic and the assumption of nearly-isotropic turbulence can no longer be maintained.

- Even when turbulence decays for an increasing stable stratification, additional turbulence is generated by the shear and breaking of internal waves which are not resolved by the model.

In the absence of a comprehensive theory of both effects, simple parameterisations have been designed and implemented in the COHERENS code. The methods consist in adding background mixing coefficients to the ones calculated by the RANS scheme. Different schemes are available.

The simplest case are uniform background coefficients for momentum and scalars, selected by the user. The program allows to reset the background value for momentum to the kinematic viscosity ν which is either a user defined or given as function of temperature using the ITTC (1978) expresssion (7.24).

The second one is based on limiting conditions for turbulence variables. The limitation of vertical overturns by stable stratification is expressed by a limiting condition of the form

$$L_T/L_O < R_l \tag{4.218}$$

where the constant R_l is of $\mathcal{O}(1)$,

$$L_T = -\sqrt{\overline{\beta^2}}/N^2 \tag{4.219}$$

is the Thorpe scale measuring the extent of the vertical excursions, and

$$L_O = (\varepsilon/N^3)^{1/2} \tag{4.220}$$

the Ozmidov scale at which buoyancy and inertial forces are of comparable magnitude. From theory, laboratory and measurements at sea (see Luyten *et al.*, 2002, and references therein), it is found that $R_l \simeq 1 - 1.3$. Using the last equation of (4.178), (4.179), (4.181) one obtains

$$\left(\frac{L_T}{L_O}\right)^2 = \frac{\overline{\beta^2}}{N\varepsilon} = 2R_\beta S_b \alpha_N^{3/2} \tag{4.221}$$

Assuming a state of quasi-equilibrium, it can be shown that the right hand side of (4.221) is an increasing function of α_N. The upper limit in (4.218) then implies a maximum value α_{max} for α_N. Its value is obtained by solving

$$2R_\beta S_b \alpha_N^{3/2} = R_l^2 = 2R_\beta R_\star \tag{4.222}$$

for $Z = \alpha_N^{1/2}$ after substitution of (4.187) or (4.190). The result is the polynomial equation

$$C_{b5}Z^3 - R_\star C_{b3}Z^2 - R_\star = 0 \tag{4.223}$$

if $c_{22\beta} = 0$ or

$$\begin{aligned} C_{c5}C_{c8}Z^6 \quad &+ \quad R_* C_{c5}C_{c7}Z^5 + (C_{c4}C_{c8} + R_*^2 C_{c5}^2)Z^4 + R_*(C_{c4}C_{c7} + C_{c5}C_{c6})Z^3 \\ &+ \quad 2R_*^2 C_{c4}C_{c5}Z^2 + R_* C_{c4}C_{c6}Z + R_*^2 C_{c4}^2 = 0 \end{aligned} \tag{4.224}$$

if $c_{22\beta} \neq 0$. Since S_u and S_b are decreasing functions of α_N, the stability functions are limited from below by the constants S_{ulim} and S_{blim}. The theory is illustrated in Figure 4.7 where the evolution of S_u and S_b as function of the Richardson number is shown for the simplified equilibrium method and using an upper limit for α_N.

Substituting the previous limits into the definitions of the eddy coefficients and α_N, the following limiting or background values are obtained

$$
\begin{aligned}
\nu_T > \nu_{Tlim} = S_{ulim}\alpha_{max}^{1/2}\frac{k}{N} = \nu_{0l}\frac{k}{N} \\
\lambda_T > \lambda_{Tlim} = S_{blim}\alpha_{max}^{1/2}\frac{k}{N} = \lambda_{0l}\frac{k}{N} \\
\varepsilon > \varepsilon_{lim} = \alpha_{max}^{-1/2}kN \\
l < l_{max} = \epsilon_0\alpha_{max}^{1/2}\frac{k^{1/2}}{N}
\end{aligned}
\tag{4.225}
$$

As explained in Luyten *et al.* (2002) a second limiting condition needs to be imposed for the turbulence energy

$$k > k_{lim} \tag{4.226}$$

yielding background values of the form $\nu_T \sim N^{-1}$, $\lambda_T \sim N^{-1}$ and $\varepsilon \sim N$.

The effect of the limiting condition can be further clarified by making the local equilibrium assumption (4.192). Substituting α_{max} into equation (4.194) and setting $\alpha_M = \alpha_N/Ri$, this equation can be solved for Ri yielding a second critical Richardson number $Ri_c^\star < Ri_c$. The eddy coefficients are then given by the previous closure schemes or by the background values (4.225) depending on whether Ri is lower or higher than $Ri_c^\star$. Values of the critical parameters α_{max}, ν_{0l}, λ_{0l} and $Ri_\star$ are listed in table 4.3.

The third method is a semi-empirical formulation originally proposed by Large *et al.* (1994)

$$
\begin{aligned}
\nu_{bT} &= \nu_{T0} + \nu_0^s\Big(1 - Ri/Ri_0\Big)^{p_1} \\
\lambda_{bT} &= \lambda_{T0} + \nu_0^s\Big(1 - Ri/Ri_0\Big)^{p_1}
\end{aligned}
\tag{4.227}
$$

where ν_{bT} and λ_{bT} are background mixing coefficients added to the eddy coefficients calculated by the model. The first terms on the right hand side represent mixing due to unresolved internal shear, the second one mixing due to internal wave braking. The parameters have the following default values

$$Ri_0 = 0.7\,,\quad p_1 = 3\,,\quad (\nu_0^s, \nu_{T0}, \lambda_{T0}) = (0.005, 10^{-4}, 0.5\times 10^{-4})\ \mathrm{m^2/s} \tag{4.228}$$

Besides the physical limiting conditions, discussed above, the following numerical lower bounds are always imposed

$$k_{min} = 10^{-14}\,,\quad \varepsilon_{min} = 10^{-12},\quad l_{min} = 1.7\times 10^{-10} \tag{4.229}$$

The reason for adopting this lower limits is to prevent that unrealistically large eddy coefficients are created by rounding errors within the intermittent zone. With the above values and taking $S_u \sim S_b \sim 0.1$ the values of ν_T and λ_T are of the order of $\sim 10^{-17}$ which are clearly much smaller than the corresponding laminar viscosity and diffusivity coefficients.

Implementation

A RANS model is selected if iopt_vdif_coef=3. A series of additional switches are implemented to determine the specifications of the model.

iopt_kinvisc — Formulation for kinematic viscosity.

- 0: user-defined uniform value kinvisc_cst
- 1: ITTC (1978) relation (7.24)

iopt_turb_dis_bbc — Type of bottom boundary condition for the dissipation rate ε.

- 1: Neumann condition (4.353)
- 2: Dirichlet condition (4.351)

iopt_turb_dis_sbc — Type of surface boundary condition for the dissipation rate ε.

- 1: Neumann condition (4.284)
- 2: Dirichlet condition (4.281)

iopt_turb_iwlim — Type of background mixing scheme as described in Section 4.4.3.6.

- 0: using uniform background coefficients
- 1: using limiting conditions for turbulence parameters
- 2: the Large *et al.* (1994) scheme given by (4.227)–(4.228)

iopt_turb_kinvisc — Selects type of background mixing mixing.

- 0: user-defined constant value vdifmom_cst, vdifscal_cst
- 1: kinematic viscosity as selected by iopt_kinvisc

iopt_turb_lmix — Mixing length formulation as described in Section 4.4.3.5.

- 1: parabolic law (4.213)
- 2: "modified" parabolic law (4.214)
- 3: "Xing" formulation (4.215)

4: "Blackadar" asymptotic formulation (4.216)

iopt_turb_ntrans — Number of transport equations as described in Section 4.4.3.4.

0: zero-equation model (equilibrium or Mellor-Yamada level 2 method) with a mixing length selected by iopt_turb_lmix

1: turbulence energy equation with a mixing length selected by iopt_turb_lmix

2: k-ε of k-kl equation depending on the value of iopt_turb_param

iopt_turb_param — Selects type of second turbulent variable.

1: mixing length l (k-l scheme)

2: dissipation rate ε (k-ε scheme)

iopt_turb_stab_form — Selects type of stability function.

1: constant value (4.198)

2: Munk-Anderson form (4.199)

3: from RANS model as explained in Section 4.4.3.3

iopt_turb_stab_lev — Selects level for stability functions if iopt_turb_stab_form = 3.

1: quasi-equilibrium method (Section 4.4.3.3)

2: non-equilibrium method (Section 4.4.3.3)

iopt_turb_stab_mod — Selects type of closure (RANS) model.

1: MY82-model (Mellor & Yamada, 1982)

2: KC94-model (Kantha & Clayson, 1994)

3: BB95-model (Burchard & Baumert, 1995)

4: HR82-model (Hossain & Rodi, 1982)

5: CA01-model (Canuto *et al.*, 2001)

6: CA02-model (Canuto *et al.*, 2001)

iopt_turb_stab_tke — Formulation for the turbulent diffusion coefficient ν_k (or stability coefficient S_k) of turbulent energy.

1: constant value for S_k as given by equation (4.200)

2: S_k is taken as proportional to momentum stability function S_u as given by (4.201)

3: using the formulation of Daly & Harlow (1970) as given by (4.185) or (4.191) depending on the value of iopt_turb_stab_lev

iopt_turb_tke_bcc — Type of bottom boundary condition for turbulence energy.

1: Neumann condition (4.352)

2: Dirichlet condition (4.351)

iopt_turb_tke_sbc — Type of surface boundary condition for turbulence energy.

1: Neumann condition (4.283)

2: Dirichlet condition (4.281)

Table 4.4 gives an overview of all parameters used in the different schemes and their default values.

Table 4.4: Parameters used in different turbulence schemes (except those listed in Tables 4.2 and 4.3) and their default values.

name	value	unit	purpose
k-l theory			
E_1	1.8	–	constant in the shear production term of the kl-equation (4.209)
E_2	1.33	–	constant in the wall proximity term (4.210) of the kl-equation (4.209)
E_3	1.0	–	constant in the buocancy source/sink term of the kl-equation (4.209)
k-ε theory			
$c_{1\varepsilon}$	1.44	–	constant in the shear production term of the ε-equation (4.205)
$c_{2\varepsilon}$	1.92	–	constant in the dissipation term of the ε-equation (4.205)
$c_{3\varepsilon}$	0.2	–	constant in the buoyancy sink term of the ε-equation (4.205) in case of stable stratification ($N^2 > 0$)
$c_{3\varepsilon}$	1.0	–	constant in the buoyancy source term of the ε-equation (4.205) in case of unstable stratification ($N^2 < 0$)
diffusion coefficients for turbulence variables			
c_{sk}	0.15	–	Daly-Harlow parameter in (4.177)
S_{k0}	0.09	–	neutral value of the stability coefficient S_k in the k-ε model (see equation (4.200))

(Continued)

Table 4.4: Continued

S_q	0.2	–	used to determine S_{k0} in the Mellor-Yamada model (see equation (4.202))
σ_k	1.0	–	used to define S_k in (4.201)
σ_{kl}	1.83	–	ratio of the diffusion coefficients in the k- and kl-equations, calculated from (4.211)
σ_ε	1.01	–	ratio of diffusion coefficients in the k- and ε-equations, calculated from (4.206)
limiting conditions			
k_{lim}	10^{-6}	J/kg	background limit for k (see equation (4.226))
k_{min}	10^{-14}	J/kg	numerical lower limit for k
l_{min}	1.7×10^{-10}	m	numerical lower limit for l
ε_{min}	10^{-12}	W/kg	numerical lower limit for ε
background mixing			
Ri_0	0.7	–	critical Richardson number in the Large *et al.* (1994) background mixing scheme (4.227)
λ_{T0}	5×10^{-5}	m^2/s	internal wave breaking diffusion coefficient for scalars in the Large *et al.* (1994) background mixing scheme (4.227)
ν_{T0}	10^{-4}	m^2/s	internal wave breaking diffusion coefficient for momentum in the Large *et al.* (1994) background mixing scheme (4.227)
ν_0^s	0.005	m^2/s	maximum mixing due to unresolved vertical shear in the Large *et al.* (1994) background mixing scheme (4.227)
boundary conditions			
c_w	0.0	–	surface wave factor used in the surface flux condition (4.283) for turbulent energy
z_{0b}	0.0	m	bottom roughness length in the mixing length formulation (4.212)
z_{0s}	0.0	m	surface roughness length in the mixing length formulation (4.212)
mixing length formulations			
α_1	0.2	–	constant in the Blackadar (1962) mixing length formulation (4.217)
β_1	2.0	–	attenuation factor in the Xing & Davies (1996) mixing length formulation (4.215)
algebraic schemes			
n_p	2.0	–	Pacanowski & Philander (1981) scheme (4.132)–(4.134)
α_p	5.0	–	Pacanowski & Philander (1981) scheme (4.132)–(4.134)
λ_{bp}	10^{-5}	m^2/s	Pacanowski & Philander (1981) scheme (4.132)–(4.134)
ν_{bp}	10^{-4}	m^2/s	Pacanowski & Philander (1981) scheme (4.132)–(4.134)

(Continued)

Table 4.4: Continued

ν_{max}	3.0	–	Pacanowski & Philander (1981) scheme (4.132)–(4.134)
ν_{0p}	0.01	m^2/s	Pacanowski & Philander (1981) scheme (4.132)–(4.134)
n_1	0.5	–	Munk & Anderson (1948) scheme (4.136)–(4.139)
n_2	1.5	–	Munk & Anderson (1948) scheme (4.136)–(4.139)
α_m	10.0	–	Munk & Anderson (1948) scheme (4.136)–(4.139)
β_m	3.33	–	Munk & Anderson (1948) scheme (4.136)–(4.139)
λ_{max}	4.0	–	Munk & Anderson (1948) scheme (4.136)–(4.139)
ν_{max}	3.0	–	Munk & Anderson (1948) scheme (4.136)–(4.139)
ν_{0m}	0.06	m^2/s	Munk & Anderson (1948) scheme (4.136)–(4.139)
C_ν	2.0	–	see equation (4.151)
K_1	0.0025	–	see equations (4.148) and (4.150)
K_2	2×10^{-5}	–	see equation (4.149)
r_1	1.0	–	see equation (4.143)
r_2	1.0	–	see equation (4.143)
δ_1	0.0	–	see equation (4.143)
δ_2	0.0	–	see equation (4.143)
$\lambda_\star$	0.0	m	see equation (4.146)
ω_1	10^{-4}	s^{-1}	see equation (4.151)

4.5 Astronomical tidal force

Tides are generated by the combined gravitational attraction of the sun and the moon. The total force is calculated as the gradient of the tidal potential Φ_{tid}. The potential can be written as a series of tidal harmonics

$$\begin{aligned}
\frac{\Phi_{tid}}{g} &= \zeta_e = (\frac{3}{2}\cos^2\phi - 1)\sum_{n=1}^{N_0} A_{0n}(t)\cos\Big(V_{0n}(t) + u_{0n}(t)\Big) \\
&\quad + \sin 2\phi \sum_{n=1}^{N_1} A_{1n}(t)\cos\Big(\lambda + V_{1n}(t) + u_{1n}(t)\Big) \\
&\quad + \cos^2\phi \sum_{n=1}^{N_2} A_{2n}(t)\cos\Big(2\lambda + V_{2n}(t) + u_{2n}(t)\Big) \\
&\quad + \cos^3\phi \sum_{n=1}^{N_3} A_{3n}(t)\cos\Big(3\lambda + V_{3n}(t) + u_{3n}(t)\Big) \\
&= \sum_{q=0}^{3} G_q(\phi) \sum_{n=1}^{N_q} A_{qn}(t)\cos\Big(q\lambda + V_{qn}(t) + u_{qn}(t)\Big) \qquad (4.230)
\end{aligned}$$

where ζ_e is the so-called equilibrium tide, i.e. the change of sea level due to the tidal attraction only, i.e. in absence of all other forces (pressure, Coriolis, ...), N_0, N_1, N_2, N_3 are the number of respectively long-period, diurnal, semi-diurnal and terdiurnal tides and q is the species index. Higher order harmonics can shown to be negligible and are therefore not included in the expansion. Once the tidal potential is known, the components of the tidal force in the momentum equations are obtained using

$$\begin{aligned} F_1^t &= \frac{g}{h_1}\frac{\partial \zeta_e}{\partial \xi_1} = \frac{g}{h_1}\left(\frac{\partial \zeta_e}{\partial \lambda}\frac{\partial \lambda}{\partial \xi_1} + \frac{\partial \zeta_e}{\partial \phi}\frac{\partial \phi}{\partial \xi_1}\right) \\ F_2^t &= \frac{g}{h_2}\frac{\partial \zeta_e}{\partial \xi_2} = \frac{g}{h_2}\left(\frac{\partial \zeta_e}{\partial \lambda}\frac{\partial \lambda}{\partial \xi_2} + \frac{\partial \zeta_e}{\partial \phi}\frac{\partial \phi}{\partial \xi_2}\right) \end{aligned} \tag{4.231}$$

The tidal amplitudes are the product of three factors

$$A_{qn}(t) = a_{qn}\alpha_{qn}f_{qn}(t) \quad 0 \leq q \leq 3 \tag{4.232}$$

The first factor a_{qn} is the astronomical tidal amplitude due to the lunar and solar attractive forces. The Earth can be considered as an elastic body and is deformed by the tidal force as well. The effect of the Earth tide is to reduce the astronomical tide and is represented by the second factor α_{qn}. Values are taken from Foreman *et al.* (1993). The third term is the so-called nodal factor and arises from modulations of the the lunar orbit. The term is always close to 1 and varies with a period of 18.6 years.

The tidal phases are the sum of the geographical factor $q\lambda$, the astronomical argument V_{qn} and the nodal correction factor u_{qn} (further discussed below). The astronomical phase is evaluated at the longitude of Greenwich ($\lambda = 0$). Its time dependence is given by the astronomical ephemerides[8]

$$V_{qn}(t) = i\tau + js + kh + lp + mN + np_s \tag{4.233}$$

where $i = q$ and j, k, l, m, n are integers, called the Doodson numbers (Doodson, 1921), characterising the constituent, and

τ the mean lunar time

s the mean longitude of the moon

h the mean longitude of the sun

p the mean longitude of the lunar perigee

[8]An additional phase lag of $\pm 90^0$ has to be added for diurnal tides and 0^0 or 180^0 for diurnal tides.

N the negative mean longitude of the ascending lunar node

p_s the mean longitude of the solar perigee

Since the six astronomical parameters are almost linear time, one can write

$$V_{qn}(t) \simeq V_{qn}(t_0) + (t - t_0)\omega_{qn} \tag{4.234}$$

where t_0 is the time of the previous mid-night at Greenwich[9] and ω_{qn} the frequency of the tidal constituent. The mean lunar time τ is related to the mean solar time (H) by

$$\tau = H - s + h \tag{4.235}$$

From (4.233) one obtains therefore

$$V_{qn}(t_0) = (j - i)s_0 + (k + i)h_0 + lp_0 + mN_0 + np_{s0} \tag{4.236}$$

$$\omega_{qn} = i\dot{\tau}_0 + j\dot{s}_0 + k\dot{h}_0 + l\dot{p}_0 + m\dot{N}_0 + n\dot{p}_{s0} \tag{4.237}$$

where a $\dot{}$ (dot) represents a time derivative and the subscript $_0$ evaluation at midnight (GMT). The astronomical ephemerides are calculated in time using the reference values at the first of January 0h (GMT) of the year 1900. Explicit expressions (in degrees), taken from Kantha & Clayson (2000b), are

$$\begin{aligned}
s_0 &= 270.434358 + 481267.88314137T - 0.001133T^2 + 1.9.10^{-6}T^3 \\
h_0 &= 279.69668 + 36000.7689925485T + 3.03.10^{-4}T^2 \\
p_0 &= 334.329653 + 4069.0340329575T - 0.10325T^2 - 1.2.10^{-5}T^3 \\
N_0 &= -259.16 + 1934.14T - 0.0021T^2 \\
p_{s0} &= 281.22083 + 1.71902T + 0.00045T^2 + 3.0.10^{-6}T^3
\end{aligned} \tag{4.238}$$

The number of Julian centuries T is given as function of the current year y and the day number within the current year d (between 1 and 366) and the current year y:

$$\begin{aligned}
T &= (27393.500528 + 1.0000000356D)/36525 \\
D &= d + 365(y - 1975) + \mathrm{INT}(y - 1973)/4
\end{aligned} \tag{4.239}$$

if the current year is 1975 or later, or

$$\begin{aligned}
T &= (0.5 + D)/36525 \\
D &= d + 365(y - 1900) + \mathrm{INT}(y - 1901)/4
\end{aligned} \tag{4.240}$$

[9]The time t must be referenced with respect to Greenwich mean time (GMT). An automatic conversion is made by the program if needed.

for years before 1975. The rate of change of the astronomical ephemerides are obtained from the above equations (e.g. Schureman, 1941)

$$\begin{aligned} \dot{\tau}_0 &= 14.492052^0/\mathrm{h}\,, \quad \dot{s}_0 = 0.549017^0/\mathrm{h}\,, \quad \dot{h}_0 = 0.041068^0/\mathrm{h} \\ \dot{p}_0 &= 0.004642^0/\mathrm{h}\,, \quad \dot{N}_0 = -0.002206^0/\mathrm{h}\,, \quad \dot{p}_{s0} = 0.000002^0/\mathrm{h} \end{aligned} \qquad (4.241)$$

from which the frequencies ω_{qn} are obtained using (4.237).

A total of 56 astronomical tidal constituents are defined within the program. The user is free to select a subset of these frequencies as part of the model setup. The characteristics of all constituents (name, Doodson numbers, frequency, amplitude, Greenwich phase) are given in Table 4.5.

Besides the "main" astronomical constituents, defined in Table 4.5, the harmonic expansion of the tidal potential shows a large number of additional harmonics with frequencies close to some "main" frequency, but with amplitudes much smaller than the main constituent. Their effect is taken into account through the nodal amplitude and phase factors f_{qn} and u_{qn}. They are determined as follows. Let

$$\zeta_{en} = a_{n0}\cos\theta + \sum_{k=1}^{N} a_{nk}\cos(\theta + \Delta\theta_k) \qquad (4.242)$$

be a cluster of constituents around the main component "n" (after omission of the common factor G_q), with amplitude a_{n0} and total phase θ. Setting $\varepsilon = a_{nk}/a_{n0} \ll 1$, one obtains

$$\begin{aligned} \zeta_{en} &= a_{n0}\Big(\cos\theta + \sum_k \varepsilon_k \cos(\theta + \Delta\theta_k)\Big) \\ &= a_{n0}\Big(\cos\theta + \sum_k \varepsilon_k(\cos\theta\cos\Delta\theta_k - \sin\theta\sin\Delta\theta_k)\Big) \\ &= a_{n0}\Big(\cos\theta(1 + \sum_k \varepsilon_k\cos\Delta\theta_k) - \sin\theta\sum_k \varepsilon_k\sin\Delta\theta_k\Big) \\ &= a_{n0} f_n \cos(\theta + u_n) \end{aligned} \qquad (4.243)$$

Defining

$$\rho_1 = 1 + \sum_k \varepsilon_k\cos\Delta\theta_k\,, \quad \rho_2 = \sum_k \varepsilon_k\sin\Delta\theta_k \qquad (4.244)$$

the nodal factors are then given by

$$f_n = \sqrt{\rho_1^2 + \rho_2^2}\,, \quad u_n = \arctan(\rho_2/\rho_1) \qquad (4.245)$$

and making a first order Taylor expansion with respect to ε_k. Values of a_{nk} are taken from the tables given by Cartwright & Tayler (1971); Cartwright & Edden (1973).

When the tidal forcing is included in the momentum equations, the tidal solutions for currents and elevations contain additional higher frequencies components. These so-called "overtides" are produced by non-linearities in the model equations and do not appear in the expansion of the tidal potential. For applications in shelf seas, where the astronomical force becomes negligible compared to the bottom stress, the tidal forcing is introduced as an open boundary condition for currents and/or elevations or as a surface boundary condition in case of a water column application. The external forcing is usually presented by an expansion of tidal harmonics which may include overtides. A list of the most relevant overtides is given Table 4.6.

Table 4.5: Doodson numbers, frequencies (degrees/h), amplitudes (cm) and Greenwich arguments (degrees) of the tidal constituents which can be used for astronomical and open boundary forcing.

Name	Doodson numbers						ω_{qn}	a_{qn}	α_{qn}	$V_{qn}(t_0)$
	i	j	k	l	m	n				
							Long-period tides (q=0)			
S_0	0	0	0	0	0	0	0.0000000	19.8419	0.693	0.0
Sa	0	0	1	0	0	-1	0.0410667	0.3103	0.693	$h - p_s$
Ssa	0	0	2	0	0	0	0.0821373	1.9542	0.693	$2h$
058	0	0	3	0	0	-1	0.123204	0.1142	0.693	$3h - p_s$
Msm	0	1	-2	1	0	0	0.4715211	0.4239	0.693	$s - 2h - p$
Mm	0	1	0	-1	0	0	0.5443747	2.2191	0.693	$s - p$
Msf	0	2	-2	0	0	0	1.0158958	0.3677	0.693	$2s - 2h$
Mf	0	2	0	0	0	0	1.0980331	4.2016	0.693	$2s$
083	0	3	-2	1	0	0	1.5695541	0.1526	0.693	$3s - 2h + p$
Mt	0	3	0	-1	0	0	1.6424078	0.8049	0.693	$3s - p$
093	0	4	-2	0	0	0	2.1139288	0.1287	0.693	$4s - 2h$
095	0	4	0	-2	0	0	2.1867825	0.1066	0.693	$4s - 2p$
							Diurnal tides (q=1)			
α_1	1	-4	2	1	0	0	12.3827651	0.0749	0.693	$-5s + 3h + p - 90^0$
$2Q_1$	1	-3	0	2	0	0	12.8542862	0.2565	0.693	$-4s + h + 2p - 90^0$
σ_1	1	-3	2	0	0	0	12.9271398	0.3098	0.693	$-4s + 3h - 90^0$
Q_1	1	-2	0	1	0	0	13.3986609	1.9387	0.6946	$-3s + h + p - 90^0$
ρ_1	1	-2	2	-1	0	0	13.4715145	0.3685	0.6948	$-3s + 3h - p - 90^0$
O_1	1	-1	0	0	0	0	13.9430356	10.1266	0.695	$-2s + h - 90^0$
τ_1	1	-1	2	0	0	0	14.0251729	0.1325	0.6956	$-2s + 3h + 90^0$
β_1	1	0	-2	1	0	0	14.4145567	0.0749	0.693	$-s - h + p + 90^0$
M_1	1	0	0	1	0	0	14.4966939	0.7965	0.6962	$-s + h + p + 90^0$
χ_1	1	0	2	-1	0	0	14.5695476	0.1522	0.6994	$-s + 3h - p + 90^0$
π_1	1	1	-3	0	0	1	14.9178647	0.2754	0.7027	$-2h + p_s - 90^0$
P_1	1	1	-2	0	0	0	14.9589314	4.7129	0.7059	$-h - 90^0$
S_1	1	1	-1	0	0	1	15.0000000	0.1116	0.7126	$p_s + 90^0$
K_1	1	1	0	0	0	0	15.0410686	14.2408	0.7364	$h + 90^0$
ψ_1	1	1	1	0	0	-1	15.0821353	0.1132	0.5285	$2h - p_s + 90^0$
ϕ_1	1	1	2	0	0	0	15.1232059	0.2028	0.6657	$3h + 90^0$
θ_1	1	2	-2	1	0	0	15.5125897	0.1526	0.6784	$s - h + p + 90^0$
J_1	1	2	0	-1	0	0	15.5854433	0.7965	0.6911	$s + h - p + 90^0$
SO_1	1	3	-2	0	0	0	16.0569644	0.1321	0.693	$2s - h + 90^0$
OO_1	1	3	0	0	0	0	16.1391017	0.4361	0.6925	$2s + h + 90^0$

(Continued)

Table 4.5: Continued

KQ_1	1	4	0	-1	0	0	16.6834764	0.0834	0.693	$3s+h-p+90^0$
Semi-diurnal tides (q=2)										
OQ_2	2	-3	0	3	0	0	27.3509801	0.0695	0.693	$-5s+2h+3p$
ε_2	2	-3	2	1	0	0	27.4238337	0.1804	0.693	$-5s+4h+p$
$2N_2$	2	-2	0	2	0	0	27.8593548	0.6184	0.693	$-4s+2h+2p$
μ_2	2	-2	2	0	0	0	27.9682084	0.7463	0.693	$-4s+4h$
238	2	-2	3	0	0	-1	28.0092751	0.0502	0.693	$-4s+5h-p_s$
244	2	-1	-1	1	0	1	28.3986628	0.0394	0.693	$-3s+h+p+p_s+180^0$
N_2	2	-1	0	1	0	0	28.4397295	4.6735	0.693	$-3s+2h+p$
246	2	-1	1	1	0	-1	28.4807962	0.0436	0.693	$-3s+3h+p-p_s$
ν_2	2	-1	2	-1	0	0	28.5125831	0.8877	0.693	$-3s+4h-p$
248	2	-1	3	-1	0	-1	28.5536498	0.0409	0.693	$-3s+5h-p-p_s$
γ_2	2	0	-2	2	0	0	28.9112506	0.0734	0.693	$-2s+2p+180^0$
H_1	2	0	-1	0	0	1	28.9430375	0.0842	0.693	$-2s+h+p_s+180^0$
M_2	2	0	0	0	0	0	28.9841042	24.4102	0.693	$-2s+2h$
H_2	2	0	1	0	0	-1	29.0251709	0.0746	0.693	$-2s+3h-p_s$
λ_2	2	1	-2	1	0	0	29.4556253	0.18	0.693	$-s+p+180^0$
L_2	2	1	0	-1	0	0	29.5284789	0.6899	0.693	$-s+2h-p+180^0$
T_2	2	2	-3	0	0	1	29.9589333	0.6636	0.693	$-h+p_s$
S_2	2	2	-2	0	0	0	30.0000000	11.3572	0.693	0.0
R_2	2	2	-1	0	0	-1	30.0410667	0.095	0.693	$h-p_s+180^0$
K_2	2	2	0	0	0	0	30.0821373	3.0875	0.693	$2h$
η_2	2	3	0	-1	0	0	30.6265120	0.1727	0.693	$s+2h-p$
295	2	4	0	0	0	0	31.1801703	0.0452	0.693	$2s+2h$
Ter-diurnal tides (q=3)										
M_3	3	0	0	0	0	0	43.4761563	0.3455	0.693	$-3s+3h$

Implementation

The following switches are available

iopt_astro_tide Disables (0) or enables (1) the inclusion of the tidal force in the momentum equations.

iopt_astro_pars Selects the type of astronomical forcing (tidal force and at open boundaries)

0: The astronomical argument is set to zero, the nodal factors are set to 1.

1: The astronomical arguments are calculated from (4.236) at Greenwich or at a user-defined reference longitude, the nodal factors are set to 1.

2: The astronomical arguments are determined from (4.236) at Greenwich or at a user-defined reference longitude, the nodal factors are calculated as function of the astronomical ephemerides

Table 4.6: Doodson numbers, origin, frequencies (degrees/h) and Greenwich arguments (degrees) of the overtides which can be used for the open boundary forcing.

Name	Doodson numbers						source	ω_{qn}	$V_{qn}(t_0)$
	i	j	k	l	m	n			
Semi-diurnal tides (q=2)									
$2SM_2$	2	4	-4	0	0	0	$2S_2 - M_2$	31.0158958	$2s - 2h$
Ter-diurnal tides (q=3)									
$2MK_3$	3	-1	0	0	0	0	$2M_2 - K_1$	42.9271398	$-4s+3h-90^0$
SO_3	3	1	-2	0	0	0	$S_2 + O_1$	43.9430356	$-2s + h - 90^0$
MK_3	3	1	0	0	0	0	$M_2 + K_1$	44.0251729	$-2s+3h+90^0$
SK_3	3	3	-2	0	0	0	$S_2 + K_1$	45.0410686	$h + 90^0$
Quarter-diurnal tides (q=4)									
MN_4	4	-1	0	1	0	0	$M_2 + N_2$	57.4238337	$-5s + 4h$
M_4	4	0	0	0	0	0	$2M_2$	57.9682084	$-4s + 4h$
MS_4	4	2	-2	0	0	0	$M_2 + S_2$	58.9841042	$-2s + 2h$
MK_4	4	2	0	0	0	0	$M_2 + K_2$	59.0662415	$-2s + 4h$
S_4	4	4	-4	0	0	0	$2S_2$	60.0000000	0.0
Sixth-diurnal tides (q=6)									
$2MN_6$	6	1	0	1	0	0	$2M_2 + N_2$	86.4079380	$-7s + 6h + p$
M_6	6	0	0	0	0	0	$3M_2$	86.4079380	$-6s + 6h$
MSN_6	6	1	-2	1	0	0	$M_2 + S_2 + N_2$	87.4238337	$-5s + 4h + p$
$2MS_6$	6	2	-2	0	0	0	$2M_2 + S_2$	87.9682084	$-4s + 4h$
$2SM_6$	6	4	-4	0	0	0	$2S_2 + M_2$	88.9841042	$-2s + 2h$
S_6	6	6	-6	0	0	0	$3S_2$	90.0000000	0.0
Eighth-diurnal tides (q=8)									
M_8	8	0	0	0	0	0	$4M_2$	115.9364169	$-8s + 8h$
$2MSN_8$	8	1	-2	1	0	0	$2M_2+S_2+N_2$	116.4079380	$-7s + 6h + p$
$3MS_8$	8	2	-2	0	0	0	$3M_2 + S_2$	116.9523127	$-6s + 6h$
$2(MS)_8$	8	4	-4	0	0	0	$2M_2 + 2S_2$	117.9682084	$-4s + 4h$
S_8	8	8	-8	0	0	0	$4S_2$	120.0000000	0.0

4.6 Solar radiation

Deriving a suitable expression for the solar radiation flux is not straightforward in view of its dependence on atmospheric parameters (atmospherical absorption and reflection, cloud coverage, albedo of the sea surface) whose influence is difficult to parameterise. The approach, described here, partially follows Rosati & Miyakoda (1988). The radiation entering at the top of the atmosphere is given by

$$Q_s = Q_0 p_{cor} H(\sin\gamma_\odot) \tag{4.246}$$

where $Q_0 = 1367.0$ W/m^2 is the solar constant, $\gamma_\odot$ the altitude of the sun and H the Heaviside function ($H(x) = 0$ for $x < 0$ and $= x$ otherwise). The factor p_{cor} represents a correction term due to the elliptical orbit of the earth and is usually expressed as a function of the day number of the year J:

$$p_{cor} = 1 + 0.03344\cos(J' - 2.8^0) \tag{4.247}$$

$$J' = 0.9856J \tag{4.248}$$

The altitude of the sun is calculated from

$$\sin\gamma_\odot = \sin\phi\sin\delta_\odot + \cos\phi\cos\delta_\odot\cos H_\odot \tag{4.249}$$

where $\delta_\odot$ is the declination of the sun, $H_\odot$ the sun's hour angle and ϕ the latitude. The angle $\delta_\odot$, measured in degrees, is obtained from the series expansion

$$\delta_\odot = \delta_0 + \sum_{n=1}^{3}(a_n\cos nJ' + b_n\sin nJ') \tag{4.250}$$

with

$$\begin{aligned}\delta_0 &= 0.33281\\ (a_1,a_2,a_3) &= (-22.984,-0.34990,-0.13980)\\ (b_1,b_2,b_3) &= (3.7872,0.03205,0.07187)\end{aligned} \tag{4.251}$$

The sun's hour angle, measured in hours, is computed by

$$H_\odot = t_h - 12 + TE + \lambda_h \tag{4.252}$$

where t_h is the hour of the day, λ_h the longitude (expressed in hours) and TE the equation of time which can be written as

$$TE = \sum_{n=1}^{3}(c_n\cos nJ' + d_n\sin nJ') \tag{4.253}$$

with

$$(c_1, c_2, c_3) = (0.0072, -0.0528, -0.0012)$$
$$(d_1, d_2, d_3) = (-0.1229, -0.1565, -0.0041) \tag{4.254}$$

Note that t_h must be given in GMT.

Taking account of absorption by the atmosphere, the direct solar radiation incident on the ocean surface, is given by

$$Q_{dir} = Q_s e^{-\tau} \tag{4.255}$$

The following form, proposed by Dogniaux (1984a,b), is considered for the extinction factor

$$\tau = m_o \delta_R t_L \tag{4.256}$$

The optical air mass m_o, Rayleigh's optical thickness δ_R and Linke's factor t_L are expressed as function of the solar altitude $\gamma_\odot$ in degrees, according to

$$\delta_R = (0.9m_o + 9.4)^{-1} \tag{4.257}$$

$$t_L = 0.021\gamma_\odot + 3.55 \tag{4.258}$$

$$m_o = [\sin\gamma_\odot + 0.15(\gamma_\odot + 3.885)^{-1.253}]^{-1} \tag{4.259}$$

The formulation, given by (4.257)–(4.259), has the advantage that it does not diverge at low solar altitudes.

The direct component of solar radiation must be supplemented by the diffuse sky radiation Q_{dif}. Following Rosati & Miyakoda (1988) it is assumed that one half of the scattered radiation reaches the ocean surface so that

$$Q_{dif} = \Big((1 - A_\alpha)Q_s - Q_{dir}\Big)/2 \tag{4.260}$$

They considered the value of 0.09 for the water vapor and ozone absorption coefficient A_α. The total radiation flux at the ocean surface under clear sky conditions is then given by

$$Q_{cs} = Q_{dir} + Q_{dif} = \frac{1}{2}Q_s(e^{-\tau} + 1 - A_\alpha) \tag{4.261}$$

The clear sky value (4.261) must be corrected for cloud coverage and reflection by the ocean surface. The empirical formula, derived by Reed (1977), appears to have a better agreement with observational data compared to other formulations (Katsaros, 1990). The short-wave radiation flux at the sea surface then finally takes the form

$$Q_{rad} = Q_{cs}(1 - 0.62f_c + 0.0019\gamma_{\odot,max})(1 - A_s) \tag{4.262}$$

where $\gamma_{\odot,max}$ is the solar altitude at noon. A constant value of 0.06 is assumed for the sea surface albedo A_s. Variations of the albedo as function of the solar altitude and atmospheric transmittance have been tested using the empirical fits derived by Payne (1972). No appreciable difference was found with the formulation (4.262). The only parameter, which needs to be supplied externally, is the fractional cloud cover f_c also used in the expression (4.275) for the long-wave radiation flux.

4.7 Surface boundary conditions

4.7.1 General form

Most of the surface boundary conditions discussed in this section are Neumann type conditions for the surface fluxes and can be written into one of the two following general forms

- A prescribed (upwards) surface flux F_s^ψ

$$\frac{\lambda_T^\psi}{h_3}\frac{\partial\psi}{\partial s} = F_s^\psi \tag{4.263}$$

- A surface flux describing the transfer across the surface

$$\frac{\lambda_T^\psi}{h_3}\frac{\partial\psi}{\partial s} = C_s^\psi(\psi_s^e - \psi_s^i) \tag{4.264}$$

  where C_s is the transfer rate (with the dimension of a velocity) and ψ_s^e, ψ_s^i are the values of ψ just above and below the surface.

A second form of surface boundary condition is the Dirichlet type where the value of ψ at the surface or at the first interior grid point is specified. Examples are the conditions (4.281) for turbulence.

Note that when the model equations are given in depth-averaged mode (Section 4.3.2), the surface boundary condition enters as an additional flux (source or sink) term in the transport equations. It is obvious that in that case only a Neumann flux conditions is allowed.

4.7.2 Currents

The surface condition for the horizontal current is as usual obtained by specifying the surface stress as function of the wind components

$$\rho_0\frac{\nu_T}{h_3}\Big(\frac{\partial u}{\partial s},\frac{\partial v}{\partial s}\Big) = (\tau_{s1},\tau_{s2}) = \rho_a C_{ds} W_{10}(U_{10},V_{10}) \tag{4.265}$$

where (U_{10}, V_{10}) are the components of the wind vector at a reference height of 10 m, $W_{10} = (U_{10}^2 + V_{10}^2)^{1/2}$ is the wind speed, $\rho_a = 1.2$ kg/m^3 the air density and C_{ds} the surface drag coefficient discussed in Section 4.8. The boundary condition for the transformed vertical velocity takes the simple form

$$\omega = 0 \tag{4.266}$$

4.7.3 Temperature

The surface boundary condition for temperature can either be taken as a Dirichlet condition in which case T_s is specified directly at (or near) the surface or a Neumann condition in which case the surface flux of temperature is given as

$$\frac{\rho_0 c_p}{h_3}\lambda_T\frac{\partial T}{\partial s} = Q_s \tag{4.267}$$

where Q_s is the downwards directed heat flux at the surface and c_p the specific heat of seawater at constant pressure. The net total heat flux into the water column is composed of a term $-Q_{nsol}$ of all non-solar contributions plus the radiative flux Q_{rad}. Only the former contributes to the surface flux of temperature, since solar radiance is absorbed within the water column.

The (upwards directed) non-solar heat flux has three components, i.e.

$$Q_{nsol} = Q_{la} + Q_{se} + Q_{lw} \tag{4.268}$$

where Q_{la} is the latent heat flux released by evaporation, Q_{se} the sensible heat flux due to the turbulent transport of temperature across the air/sea interface and Q_{lw} the net long-wave radiation emitted at the sea surface. The first two terms are related to the turbulent fluxes of humidity and temperature

$$Q_{la} = \rho_a L_\nu C_e W_{10}(q_s - q_a) \tag{4.269}$$

$$Q_{se} = \rho_a c_{pa} C_h W_{10}(T_s - T_a) \tag{4.270}$$

where T_s, q_s and T_a, q_a are the temperature and specific humidity at respectively the sea surface and a reference height, usually taken at 10 m, and

$$c_{pa} = 1004.6(1 + 0.8375 q_a)\ \mathrm{J\ kg^{-1}\ (^0C)^{-1}} \tag{4.271}$$

the specific heat of air at constant pressure. The latent heat of vaporization is given as a function of the sea surface temperature

$$L_\nu = 2.5008\times10^6 - 2300 T_s\ \ \mathrm{J/kg} \tag{4.272}$$

The sea surface and air humidities q_s and q_a are calculated using

$$q = \frac{\epsilon_R e}{P_{a0} - (1-\epsilon_R)e} \tag{4.273}$$

where P_a is the atmospheric pressure (in mbar) and $\epsilon_R = 0.62197$ the ratio of molecular weight of dry water to dry air. The vapour pressure e is obtained in mb from the empirical relation (Gill, 1982)

$$\log_{10} e = \log_{10} RH + \frac{0.7859 + 0.03477T}{1 + 0.00412T} \tag{4.274}$$

where RH is the relative humidity (between 0 and 1). In equations (4.273) and (4.274) the humidity q, the vapour pressure e and the temperature T (in ^{0}C) either represent sea surface or atmospheric values at the reference height. Note that a relative humidity of 100% is taken at the sea surface.

The long-wave radiation flux term is parameterised following Gill (1982):

$$Q_{lw} = \epsilon_s \sigma_{rad}(T_s + 273.15)^4(0.39 - 0.05e_a^{1/2})(1 - 0.6f_c^2) \tag{4.275}$$

where $\epsilon_s = 0.985$ is the emissivity at the sea surface, $\sigma_{rad} = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} Stefan's constant, f_c the fractional cloud cover (between 0 and 1) and e_a the vapour pressure evaluated by (4.274).

The surface fluxes of momentum and heat involve the surface drag coefficient C_{ds} and two dimensionless parameters C_e, C_h sometimes referred as the Dalton and Stanton number. Various empirical schemes for these transfer coefficients have been presented in the literature (see e.g. Blanc, 1985; Geernaert, 1990). A few formulations are available in the program. They are further discussed in Section 4.8.

Implementation

The type of surface condition for temperature is selected with the switch iopt_temp_sbc:

1: Neumann (flux) condition

2: Dirichlet condition at the first grid point below the surface

3: Dirichlet condition at the surface itself

4.7.4 Salinity

The surface salinity flux is determined using the formula given by Steinhorn (1991):

$$\rho_0 \frac{\lambda_T}{h_3}\frac{\partial S}{\partial s} = \frac{S_s(E_{vap} - R_{pr})}{1 - 0.001S_s} \tag{4.276}$$

where $E_{vap} = Q_{la}/L_\nu$ and R_{pr} are the evaporation and precipitation rates in kg m^{-2} s^{-1} and S_s the surface salinity in PSU. The evaluation of the surface salinity flux requires the input of precipitation data as an additional meteorological parameter.

Implementation

The switch iopt_sal_sbc enables (1) or disables (0) the surface flux condition (4.276).

4.7.5 Turbulence

The surface boundary conditions for turbulence variables are derived by making the "wall"- (or "log"-) layer approximation. The following assumptions are made

1. The layer is neutrally stratified ($N^2 = 0$).

2. The shear stress is taken as vertically uniform. Neglecting the Coriolis force, taking the X-axis along the flow direction and using equation (4.179) one has

$$u_{\star s}^2 = \sqrt{\tau_{s1}^2 + \tau_{s2}^2} = -S_{u0}\frac{k^2}{\varepsilon}\frac{\partial U}{\partial z} = \text{constant} \tag{4.277}$$

3. The mixing length is proportional to the distance d from the "wall" boundary as given by equation (4.212):

$$l = l_2 = \kappa d_s = \kappa(\zeta - z + z_{0s}) = \kappa\Big(H(1-\sigma) + z_{0s}\Big) \tag{4.278}$$

   where $\kappa = 0.4$ is von Kármàn's constant and z_{0s} a surface roughness length.

4. The velocity shear is given by

$$\frac{\partial U}{\partial z} = -\frac{u_\star}{l} = -\frac{u_\star}{\kappa d_s} \tag{4.279}$$

   Integration of this equation gives the familiar linear U versus $\log z$ dependence.

5. Turbulence is assumed to be in equilibrium, i.e.

$$P = -u_{\star s}^2\frac{\partial U}{\partial z} = \varepsilon = \epsilon_0\frac{k^{3/2}}{\kappa d_s} \tag{4.280}$$

From (4.277)–(4.280) the following Dirichlet type surface conditions are derived for the k, ε and kl transport equations

$$k = \frac{u_{\star s}^2}{S_{u0}^{1/2}}, \quad \varepsilon = \frac{u_{\star s}^3}{\kappa d_s}, \quad l = \kappa d_s \tag{4.281}$$

with d_s defined through (4.278). The relation

$$\epsilon_0 = S_{u0}^{3/4} \tag{4.282}$$

can be readily obtained in addition to the previous relations.

The program allows to use Neumann type condition for k and ε as well. They are given by

$$\frac{\nu_k}{h_3}\frac{\partial k}{\partial s} = c_w u_{\star s}^3 \tag{4.283}$$

$$\frac{\nu_k}{h_3 \sigma_\varepsilon}\frac{\partial \varepsilon}{\partial s} = \frac{\nu_k}{h_3 \sigma_\varepsilon}\frac{\epsilon_0 k^{3/2}}{\kappa d_s^2} \tag{4.284}$$

The first condition was proposed by Craig & Banner (1994) with $c_w \sim 100$ and a non-zero surface roughness to represent the energy input of breaking surface waves. The second one, introduced by Burchard & Petersen (1999), can be derived from the Dirichlet conditions and has, according to these authors, a better numerical performance for applications with a coarse vertical resolution. A modification of the flux condition for ε which takes account of wave breaking, was considered by Burchard (2001) but is currently not implemented in the code.

Implementation

The surface boundary condition for turbulence are selected by the following switches:

iopt_turb_tke_sbc Type of condition for k

1: Neumann condition (4.283)

2: Dirichlet condition (4.281)

iopt_turb_dis_sbc Type of condition for ε

1: Neumann condition (4.284)

2: Dirichlet condition (4.281)

4.7.6 Water column mode

In the water column approximation (Section 4.3.1), the surface slope and elevations can be prescribed at the surface as the sum of a non-harmonic and harmonic part

$$q^e(t) = q_0^e(t) + \sum_{n=1}^{N} A_n f_n(t) \cos\Big(V_n(t) + u_n(t) - \varphi_n\Big) \tag{4.285}$$

where q^e represents the external value of $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ or ζ, (f_n, u_n) are the nodal amplitude and phase factors, $V_n(t)$ the astronomical phases at Greenwich and (A_n, φ_n) the amplitudes and phases with respect to Greenwich[10].

4.8 Surface drag and exchange coefficients

The values of C_{ds}, C_E and C_H depend on conditions in the lower atmosphere and are, in general, functions of W_{10}, T_a, T_s, RH (relative humidity) and P_a. Several (mainly empirical) formulations are implemented and can be divided into neutral schemes, depending on wind speed only, and stratified ones which take additionally account of at least the effect of the air minus sea temperature difference.

4.8.1 Neutral formulations

The following formulations for C_{ds} are implemented

0. Constant value. Default value is 0.0013.

1. Large & Pond (1981)

$$\begin{aligned} C_{ds} &= 0.0012 & \text{if} \quad & W_{10} \leq 11\text{m/s} \\ C_{ds} &= 10^{-3}(0.49 + 0.065 W_{10}) & \text{if} \quad & W_{10} > 11\text{m/s} \end{aligned} \tag{4.286}$$

2. Smith & Banke (1975)

$$C_{ds} = 10^{-3}(0.63 + 0.066 W_{10}) \tag{4.287}$$

3. Geernaert *et al.* (1986)

$$C_{ds} = 10^{-3}(0.43 + 0.097 W_{10}) \tag{4.288}$$

[10]Expression (4.285) is analogous to (4.354) applied at open boundaries.

Table 4.7: Empirical parameters used in the Kondo (1975) formulations for the neutral surface drag and exchange coefficients.

	a_d, b_d, p_d	a_e, b_e, c_e, p_e	a_h, b_h, c_h, p_h
$W_{10} < 2.2$	0.0, 1.08, -0.15	0.0, 1.23, 0.0, -0.16	0.0, 1.185, 0.0, -0.157
$2.2 \leq W_{10} < 5$	0.771, 0.0858, 1.0	0.969, 0.0521, 0.0, 1.0	0.927, 0.0546, 0.0, 1.0
$5 \leq W_{10} < 8$	0.867, 0.0667, 1.0	1.18, 0.0, 0.0, 1.0	1.15, 0.01, 0.0, 1.0
$8 \leq W_{10} < 25$	1.2, 0.025, 1.0	1.196, 0.008, -0.0004, 1.0	1.17, 0.0075, -0.00045, 1.0
$25 \leq W_{10}$	0.0, 0.073, 1.0	1.68, -0.016, 0.0, 1.0	1.652, -0.017, 0.0, 1.0

4. Kondo (1975)

$$C_{ds} = 10^{-3}(a_d + b_d W_{10}^{p_d}) \tag{4.289}$$

where a_d, b_d and p_d are function of wind speed and given in Table 4.7.

5. Wu (1980)

$$\begin{aligned} C_{ds} &= 0.0012 R_w^{0.15} \\ \log_{10} R_w &= 0.137 W_{10} - 0.616 \end{aligned} \tag{4.290}$$

6. Charnock (1955) relation

$$\begin{aligned} z_{0s} g / u_{*s}^2 &= a \\ C_{ds} &= \left(\frac{\kappa}{\ln(z_{0s}/10)}\right)^2 \end{aligned} \tag{4.291}$$

where z_{0s} is the surface roughness length, u_* the surface friction velocity and a=0.014 Charnock's constant. Since $u_{*s}^2 = C_{ds} W_{10}^2$, equation (4.291) has to be solved by iteration.

Neutral values for C_e and C_h are obtained from one of the following schemes.

0. Constant values. Default is 0.0013.

1. Large & Pond (1982)

$$\begin{aligned} C_e &= 0.00115 \\ C_h &= 0.00113 \quad \text{if} \quad T_a < T_s \\ &= 0.00066 \quad \text{if} \quad T_a \geq T_s \end{aligned} \tag{4.292}$$

2. Anderson & Smith (1981)

$$\begin{aligned} C_e &= 10^{-3}(0.55 + 0.083 W_{10}) \\ C_h &= 0.00112 \quad \text{if} \quad T_a < T_s \\ &= 0.00082 \quad \text{if} \quad T_a \geq T_s \end{aligned} \tag{4.293}$$

3. Kondo (1975)

$$\begin{aligned} C_e &= 10^{-3}\Big(a_e + b_e W_{10}^{p_e} + c_e(W_{10} - 8)^2\Big) \\ C_h &= 10^{-3}\Big(a_h + b_h W_{10}^{p_h} + c_h(W_{10} - 8)^2\Big) \end{aligned} \tag{4.294}$$

   where the coefficients a_e, a_h, b_e, b_h, c_e, c_h and p_e, p_h are function of the wind speed and given in Table 4.7.

4. Wu (1980)

$$\begin{aligned} C_e &= C_h = 0.001 R_w^{0.11} \\ \log_{10} R_w &= 0.137 W_{10} - 0.616 \end{aligned} \tag{4.295}$$

4.8.2 Kondo's stratified formulation

Kondo (1975) extended the neutral formulation (4.289), (4.294) for stratified conditions. The method consists in multiplying the neutral values by a factor depending on the air minus sea temperature difference. The procedure is as follows

$$R_0 = (T_s - T_a)/W_{10}^2 \tag{4.296}$$

$$R = R_0 \frac{|R_0|}{|R_0| + 0.01} \tag{4.297}$$

In case of stable conditions ($T_s < T_a$)

$$\begin{aligned} f(R) &= 0.1 + 0.03R + 0.9e^{4.8R} \quad \text{if} \quad -3.3 < R < 0 \\ f(R) &= 0 \quad \text{if} \quad R \leq -3.3 \end{aligned} \tag{4.298}$$

$$C_{ds} = C_{dn} f(R), \; C_e = C_{en} f(R), \; C_h = C_{hn} f(R) \tag{4.299}$$

For unstable conditions ($T_s > T_a$)

$$\begin{aligned} C_{ds} &= C_{dn}(1.0 + 0.47 R^{1/2}) \\ C_e &= C_{en}(1.0 + 0.63 R^{1/2}) \\ C_h &= C_{hn}(1.0 + 0.63 R^{1/2}) \end{aligned} \tag{4.300}$$

where C_{dn}, C_{en}, C_{hn} are the neutral values in the absence of stratification. In principle one may choose any of the previous relations for the neutral coefficients. It is however recommended to use the ones proposed by Kondo (1975).

4.8.3 Stratified case from Monin-Obukhov theory

The most general way, but also the most complex one, to include stratification is based on the Monin-Obukhov similarity theory as described in e.g. Geernaert (1990); Kantha & Clayson (2000a). Some details of the physical theory are given here to understand how it is implemented in the program.

The surface values of the momentum, latent and sensible heat fluxes at the air-sea interface depend on the turbulent structure of the lower atmosphere, usually called the planetary boundary layer. It is generally assumed that the fluxes of momentum, heat and specific humidity have nearly constant values within this layer. Following Monin & Obukhov (1954) the structure of this layer can be described by means of a velocity scale u_* (friction velocity), a temperature scale T_* and a humidity scale q_* defined by

$$u_* = (\langle u'w' \rangle^2 + \langle v'w' \rangle^2)^{1/2} \tag{4.301}$$

$$u_* T_* = -\langle w'T' \rangle \tag{4.302}$$

$$u_* q_* = -\langle w'q' \rangle \tag{4.303}$$

where (u', v', w'), T' and q' are the turbulent fluctuations of the wind velocity, temperature and humidity, and $\langle\ \rangle$ denotes an ensemble average. The vertical gradient of the wind speed U is written as the ratio of u_* to a mixing length, or

$$\frac{\partial U}{\partial z} = \frac{u_*}{l} \tag{4.304}$$

where z is the height above the sea surface. For neutral stratification one has

$$l = \kappa z \tag{4.305}$$

Since l decreases or increases with respect to its neutral value, according as the stratification is stable or unstable, equation (4.304) can be rewritten in the more general form

$$\frac{\partial U}{\partial z} = \frac{u_*}{\kappa z} \phi_m \tag{4.306}$$

The dimensionless function ϕ_m describes the effect of stratification and is smaller (or larger) than 1 for unstable (stable) stratification. In a similar way, the gradients of temperature and relative humidity are given by

$$\frac{\partial T}{\partial z} = \frac{T_*}{\kappa z} \phi_h \tag{4.307}$$

$$\frac{\partial q}{\partial z} = \frac{q_*}{\kappa z} \phi_q \tag{4.308}$$

The functions ϕ_m, ϕ_h and ϕ_q are expressed in terms of the dimensionless height $\xi = z/L_{mo}$ with the Monin-Obukhov length L_{mo} defined by

$$L_{mo}^{-1} = -\frac{g\kappa}{T_v u_*^3}(\langle w'T'\rangle + 0.61T_k\langle w'q'\rangle) \tag{4.309}$$

where T_k represents the temperature in degrees Kelvin and the virtual temperature T_v is given by

$$T_v = T_k(1 + 0.61q) \tag{4.310}$$

Note that $\xi > 0$ for stable and $\xi < 0$ for unstable stratification. Based upon atmospheric measurements (Businger *et al.*, 1971) the following parameterisations are adopted

$$\begin{aligned} \phi_m &= (1-\alpha\xi)^{-1/4} \quad \text{for} \quad \xi < 0 \\ \phi_m &= 1 + \beta\xi \quad \text{for} \quad \xi > 0 \end{aligned} \tag{4.311}$$

and

$$\begin{aligned} \phi_h &= \phi_m^2 \quad \text{for} \quad \xi < 0 \\ \phi_h &= \phi_m \quad \text{for} \quad \xi > 0 \end{aligned} \tag{4.312}$$

with $\alpha = 16$, $\beta = 5$, while it is further assumed that $\phi_q = \phi_h$. Integrating (4.306)–(4.308) one obtains

$$U = \frac{u_*}{\kappa}(\ln\frac{z}{z_{0U}} - \psi_m) \tag{4.313}$$

$$T - T_s = \frac{T_*}{\kappa}(\ln\frac{z}{z_{0T}} - \psi_h) \tag{4.314}$$

$$q - q_s = \frac{q_*}{\kappa}(\ln\frac{z}{z_{0q}} - \psi_h) \tag{4.315}$$

where

$$\psi_m = \int_0^\xi \frac{1-\phi_m(\xi)}{\xi}\,d\xi \tag{4.316}$$

$$\psi_h = \int_0^\xi \frac{1-\phi_h(\xi)}{\xi}\,d\xi \tag{4.317}$$

The subscript s indicates a quantity evaluated at the sea surface. Expressions (4.313)–(4.315) are valid for $z \gg z_{0U}, z_{0T}, z_{0q}$ while it is further assumed that $U \gg U_s$. The roughness lengths z_{0U}, z_{0T} and z_{0q} prevent the quantities becoming too large near the surface. Evaluating the integrals (4.316)–(4.317) one has

$$\begin{aligned} \psi_m &= 2\ln(1+\phi_m^{-1}) + \ln(1+\phi_m^{-2}) - 2\arctan(\phi_m^{-1}) + \tfrac{\pi}{2} - 3\ln 2 \quad \text{for} \quad \xi < 0 \\ \psi_m &= 1 - \phi_m \quad \text{for} \quad \xi > 0 \end{aligned} \tag{4.318}$$

and

$$\begin{array}{ll} \psi_h = 2\ln(1+\phi_h^{-1}) - 2\ln 2 & \text{for} \quad \xi < 0 \\ \psi_h = 1 - \phi_h & \text{for} \quad \xi > 0 \end{array} \tag{4.319}$$

Parameterisations for u_*, T_*, q_* are given by rewriting (4.313)–(4.315):

$$u_*^2 = \kappa^2 (\ln \frac{z}{z_{0U}} - \psi_m)^{-2} U^2 = C_{ds} U^2 \tag{4.320}$$

$$u_* T_* = \kappa^2 (\ln \frac{z}{z_{0U}} - \psi_m)^{-1} (\ln \frac{z}{z_{0T}} - \psi_h)^{-1} U(T - T_s) = C_h U(T - T_s) \tag{4.321}$$

$$u_* q_* = \kappa^2 (\ln \frac{z}{z_{0U}} - \psi_m)^{-1} (\ln \frac{z}{z_{0q}} - \psi_h)^{-1} U(q - q_s) = C_e U(q - q_s) \tag{4.322}$$

where C_{ds}, C_h and C_e are the drag coefficient, the Stanton and the Dalton numbers. The z-dependence of the coefficients is usually eliminated by evaluating them at the standard height $z = z_a = 10$ m. Suitable expressions are required for z_{0U}, z_{0T}, z_{0q}. Parameterisations exist for the roughness length as function of the wave state (e.g. Janssen, 1991). However, since little information is available concerning the form of z_{0T} and z_{0q}, the simpler approach described in Geernaert (1990) is used. This consists in defining a drag coefficient C_{dn} valid for a neutral stratification. In analogy with (4.320) one has

$$U = C_{dn}^{-1/2} u_{*n} = \frac{u_{*n}}{\kappa} \ln \frac{z}{z_{0n}} \tag{4.323}$$

where z_{0n} is the value of z_{0U} for neutral conditions. Eliminating z between (4.320) and (4.323) yields the following relation between C_{ds} and C_{dn}:

$$C_{ds} = [C_{dn}^{-1/2} + (\ln \frac{z_{0n}}{z_{0U}} - \psi_m)/\kappa]^{-2} \tag{4.324}$$

Following Charnock (1955) one further assumes that the roughness length scales with the wind stress or

$$z_{0U} = a u_*^2 / g \tag{4.325}$$

so that

$$\frac{z_{0U}}{z_{0n}} = \frac{u_*^2}{u_{*n}^2} = \frac{C_{ds}}{C_{dn}} \tag{4.326}$$

Hence

$$C_{ds} = [C_{dn}^{-1/2} + (\ln(C_{dn}/C_{ds}) - \psi_m)/\kappa]^{-2} \tag{4.327}$$

The neutral Stanton and Dalton numbers are defined in a similar way

$$C_{hn} = \kappa^2 (\ln \frac{z}{z_{0n}} \ln \frac{z}{z_{0T}})^{-1} \tag{4.328}$$

$$C_{en} = \kappa^2 (\ln \frac{z}{z_{0n}} \ln \frac{z}{z_{0q}})^{-1} \tag{4.329}$$

Expressions for C_h and C_e in terms of their neutral counterparts are then obtained by combining (4.328)–(4.329) with (4.321)–(4.322) (taking $z_{0U} \simeq z_{0n}$ for simplicity) and (4.323). This gives

$$C_h = C_{hn}[1 - (\psi_m C_{dn}^{1/2} + \psi_h C_{hn} C_{dn}^{-1/2})/\kappa + C_{hn}\psi_m\psi_h/\kappa^2]^{-1} \tag{4.330}$$

$$C_e = C_{en}[1 - (\psi_m C_{dn}^{1/2} + \psi_h C_{en} C_{dn}^{-1/2})/\kappa + C_{en}\psi_m\psi_h/\kappa^2]^{-1} \tag{4.331}$$

The neutral coefficients are obtained from one of the formulations given in Section 4.8.1.

Since the functions ψ_m and ψ_h depend on the dimensionless height ξ, a further equation needs to be added. Eliminating u_*, T_* and q_* in (4.309) after substituting (4.302)–(4.303), by using (4.320)–(4.322) and evaluating at the reference height, one has

$$\xi = \frac{g\kappa z_a}{T_v W_{10}^2} \frac{C_h(T_a - T_s) + 0.61 T_k C_e (q_a - q_s)}{(C_{ds})^{3/2}} \tag{4.332}$$

In summary, the coefficients C_{ds}, C_h and C_e are obtained by solving the system consisting of the four equations (4.327), (4.330), (4.331) and (4.332) using an iteration scheme. Input parameters are the wind speed W_{10}, the air temperature T_a, the sea surface temperature T_s, the relative humidity RH and the atmospheric pressure p_a . The last two are needed to evaluate q_a, q_s through (4.273)–(4.274).

The various schemes introduced in this section are compared in Figure 4.8. The following observations can be made

- The Smith & Banke (1975) and Charnock (1955) formulations are highly similar. Larger diferences between the different schemes, up to a factor 2, are seen for the surface drag coefficient in the case of high wind speeds.
- Stratification effects measured by the air-sea temperature difference are important at wind speeds below 10 m/s. A significant decrease of the exchange coefficients is observed in case of a stable ($T_a > T_s$) stratification, whereas the cofficients increase in the unstable ($T_a < T_s$) case.
- The Kondo and Monin-Obukhov formulations are qualitatively similar.
- The effect of relative humidity is less significant compared to the one produced by stratification.

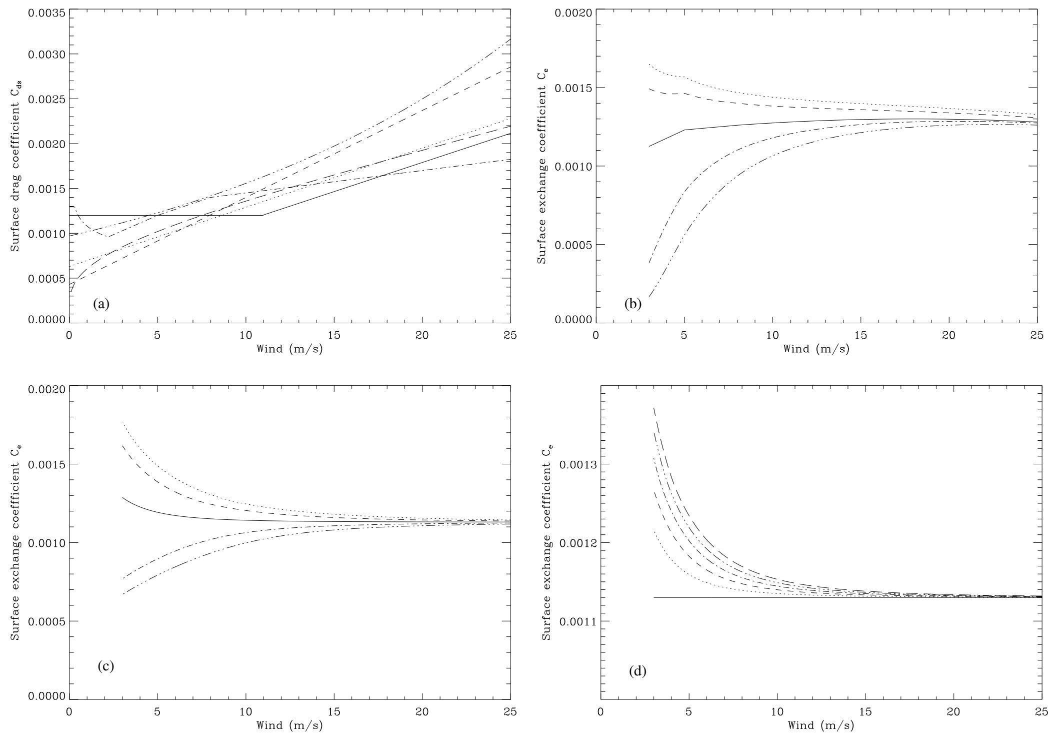

Figure 4.8: (a) Surface drag coefficient C_{ds} as function of wind speed according to (4.286) (solid), (4.287) (dots), (4.288) (dashes), (4.289) (dash-dots), (4.290) (dash and 3 dots), (4.291) (long dashes). (b) Surface exchange coefficient C_e as function of wind speed according to the Kondo (1975) formulation and $\Delta T = T_a - T_s = 0^0$C (solid), -$5^0$C (dots), -$2.5^0$C (dashes), 2.5^0C (dash-dots), 5^0C (dash and 3 dots). (c) Surface exchange coefficient C_e as function of wind speed according to Monin-Obukhov theory, using RH=75%, T_s=12^0C and $\Delta T = T_a - T_s = 0^0$C (solid), -$5^0$C (dots), -$2.5^0$C (dashes), 2.5^0C (dash-dots), 5^0C (dash and 3 dots). (d) Surface exchange coefficient C_e as function of wind speed according to Monin-Obukhov theory, using $T_a = T_s = 12^0$C and RH=100% (solid), 90% (dots), 80% (dashes), 70% (dash-dots), 60% (dash and 3 dots), 50% (long dashes).

Implementation

Evaluation of the surface drag and exchange coefficients is selected with the following switches

iopt_sflux_cds Formulation for the neutral surface drag coefficient C_{ds}.

- 0 : constant value as given by the parameter cds_cst
- 1 : equation (4.286) from Large & Pond (1981)
- 2 : equation (4.287) from Smith & Banke (1975)
- 3 : equation (4.288) from Geernaert *et al.* (1986)
- 4 : equation (4.289) from Kondo (1975)
- 5 : equation (4.290) from Wu (1980)
- 6 : equation (4.291) from Charnock (1955)

iopt_sflux_cehs Formulation for the neutral surface (heat) exchange coefficients C_e, C_h.

- 0 : constant value as given by the parameter ces_cst or chs_cst
- 1 : equation (4.292) from Large & Pond (1982)
- 2 : equation (4.293) from Anderson & Smith (1981)
- 3 : equation (4.294 from Kondo (1975)
- 4 : equation (4.295) from Wu (1980)

iopt_sflux_strat Selects dependence of the surface drag and exchange coefficients on atmospheric stratification effects.

- 0 : no dependence
- 1 : using the Kondo (1975) parameterisation (Section 4.8.2)
- 2 : using Monin-Obukhov similarity theory (Section 4.8.3)

4.9 Bottom boundary conditions

4.9.1 General form

In analogy with Section 4.7.1 most of the bottom boundary conditions are flux (Neumann type) conditions and can be written into one of the two following general forms

- A prescribed (upwards) bottom flux F_b^ψ

$$\frac{\lambda_T^\psi}{h_3}\frac{\partial\psi}{\partial s} = F_b^\psi \tag{4.333}$$

- A bottom flux describing the transfer across the sea bed

$$\frac{\lambda_T^\psi}{h_3}\frac{\partial\psi}{\partial s} = C_b^\psi(\psi_b^i - \psi_b^e) \tag{4.334}$$

  where C_b^ψ is the transfer rate (with the dimension of a velocity) and ψ_b^e, ψ_b^i are the values of ψ just below and above the sea bed.

For example, the bottom conditions (4.336) and (4.340) for u and v are of the form (4.334) with

$$C_b^u = C_b^v = C_{db}(u_b^2 + v_b^2)^{1/2}\,, \quad u_b^e = v_b^e = 0 \tag{4.335}$$

An alternative form is a Dirichlet boundary condition where the value of ψ at the bottom or the first interior point is specified. Examples are the conditions (4.351) for turbulence.

Note that when the model equations are given in depth-averaged mode (Section 4.3.2), the bottom boundary condition enters as an additional flux (source or sink) term in the transport equations. It is obvious that in that case only a Neumann flux condition is allowed.

4.9.2 Currents

A slip boundary condition is applied for the horizontal current at the bottom which takes the form

$$\frac{\nu_T}{h_3}\Big(\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s}\Big) = (\tau_{b1}, \tau_{b2}) \tag{4.336}$$

The following formulations have been implemented

- zero stress condition

$$(\tau_{b1}, \tau_{b2}) = (0, 0) \tag{4.337}$$

- linear friction law, either in 3-D as in 2-D mode

$$(\tau_{b1}, \tau_{b2}) = k_{lin}(u_b, v_b) \tag{4.338}$$

  or

$$(\tau_{b1}, \tau_{b2}) = k_{lin}(\overline{u}, \overline{v}) \tag{4.339}$$

- quadratic friction law, either in 3-D as in 2-D mode

$$(\tau_{b1},\tau_{b2}) = C_{db}(u_b^2+v_b^2)^{1/2}(u_b,v_b) \tag{4.340}$$

or

$$(\tau_{b1},\tau_{b2}) = C_{db}(\overline{u}^2+\overline{v}^2)^{1/2}(\overline{u},\overline{v}) \tag{4.341}$$

where the bottom currents (u_b, v_b) are evaluated at the grid point nearest to the bottom and $(\overline{u},\overline{v})$ are the depth-mean currents. A constant value is taken for the linear friction coefficient k_{lin}.

The quadratic law for the 3-D case is obtained using the boundary layer approximation of a vertically uniform shear stress (see equation 4.349, yielding a logarithmic profile for the current

$$|\mathbf{u}(z)| = \frac{u_{*b}}{\kappa}\ln(\frac{z_*}{z_0}) \tag{4.342}$$

where $u_{*b}^2 = \tau_b$, z_* the height above the sea bed and z_0 the bottom roughness length. The quadratic bottom drag coefficient can then be expressed as a function of the roughness length z_0 and the location of the bottom cell. This gives

$$C_{db} = \Big(\kappa/\ln(z_r/z_0)\Big)^2 \tag{4.343}$$

where z_r is a reference height taken at the grid centre of the bottom cell. The value of z_0 which may vary in the horizontal directions, depends on the geometry and composition of the seabed. Values of z_0, measured from logarithmic current profiles can be found in Heathershaw (1981); Soulsby (1983, 1997) for various bed type forms.

When COHERENS is used in 2-D mode, the drag coefficient is determined by averaging (4.342) over the water depth. Assuming $z_0 \ll H$, one obtains

$$C_{db} = \Big[\kappa/\ln\big(H/(ez_0)\big)\Big]^2 \tag{4.344}$$

Note that in depth-averaged (2-D) mode, the only allowed formulations for the bottom stress are (4.339), (4.341) and (4.344).

The log-layer approximation is only valid if $z_b \gg z_0$. This may create a problem in case the grid cell is drying and $z_b \rightarrow z_0$, $C_{db} \rightarrow \infty$. To prevent too large drag coefficients, a lower limit has been imposed of the form $z_b/z_0 > \xi_{min}$. In the previous versions this limit was set internally to a value of 1.5. In the current version ξ_{min} is user-defined. Default value is 2 yielding a maximum of 0.333 for C_{db}.

In analogy with the surface condition (4.266) the bottom value of the transformed vertical velocity equals zero, i.e.

$$\omega = 0 \tag{4.345}$$

Implementation

Evaluation of the bottom stress is controlled by the following switches

iopt_bstres_form Type of bottom stress formulation

0: Zero bottom stress
1: Linear friction law
2: Quadratic friction law

iopt_bstres_nodim Type of currents used in the bottom stress formulation

2: Depth mean currents
3: 3-D current at the bottom grid cell

iopt_bstres_drag Type of formulation for C_{db}

0: Set to zero
1: Constant prescribed value
2: Prescribed horizontally non-uniform value
3: Using (4.343) or (4.344) and a constant roughness length
4: Using (4.343) or (4.344) and a spatially dependent roughness length

4.9.3 Temperature and salinity

The bottom boundary conditions for temperature and salinity are obtained by considering a zero flux normal to the seabed:

$$\frac{\lambda_T}{h_3}\frac{\partial T}{\partial s} = 0 \ , \ \frac{\lambda_T}{h_3}\frac{\partial S}{\partial s} = 0 \tag{4.346}$$

A similar assumption is applied for the absorption term in the temperature equation (4.47) where solar radiance I is set to zero at the sea bottom. It is remarked that the non-allowance of any heat exchange at the bottom interface may not be realistic but is only imposed in the absence of a useful parameterisation which takes account of a bottom exchange (e.g. release of geothermal energy).

4.9.4 Turbulence

The bottom boundary conditions are obtained using the same boundary layer assumptions as for the surface case. Equations (4.277)–(4.280) are replaced by equations (4.347)–(4.350) below. The bottom friction velocity is defined by

$$u_{\star b}^2 = \sqrt{\tau_{b1}^2 + \tau_{b2}^2} = S_{u0}\frac{k^2}{\varepsilon}\frac{\partial U}{\partial z} = \text{constant} \tag{4.347}$$

The mixing length is proportional to the distance d_b from the "wall" boundary as given by equation (4.212):

$$l = l_1 = \kappa d_b = \kappa(h + z + z_{0b}) = \kappa\Big(H\sigma + z_{0b}\Big) \tag{4.348}$$

where is z_{0b} a bottom roughness length.

$$\frac{\partial U}{\partial z} = \frac{u_{\star b}}{l} = \frac{u_{\star b}}{\kappa d_b} \tag{4.349}$$

$$P = u_{\star b}^2\frac{\partial U}{\partial z} = \varepsilon = \epsilon_0\frac{k^{3/2}}{\kappa d_b} \tag{4.350}$$

The following Dirichlet conditions are derived from the previous equations

$$k = \frac{u_{\star b}^2}{S_{u0}^{1/2}}\,, \quad \varepsilon = \frac{u_{\star b}^3}{\kappa d_b}\,, \quad l = \kappa d_b \tag{4.351}$$

In analogy with the surface case the conditions for k and ε can be replaced by conditions for the bottom flux

$$\frac{\nu_k}{h_3}\frac{\partial k}{\partial s} = 0 \tag{4.352}$$

$$\frac{\nu_k}{h_3\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial s} = \frac{\nu_k}{h_3\sigma_\varepsilon}\frac{\epsilon_0 k^{3/2}}{\kappa d_b^2} \tag{4.353}$$

The first condition states that there is no energy flux across the bottom.

Implementation

Bottom boundary condition for turbulence are selected by the following switches:

iopt_turb_tke_bbc Type of condition for k

1: Neumann condition (4.352)

2: Dirichlet condition (4.351)

iopt_turb_dis_bbc Type of condition for ε

1: Neumann condition (4.353)

2: Dirichlet condition (4.351)

4.10 Lateral boundary conditions

4.10.1 Open boundary conditions for the 2-D mode

The model uses a Arakawa C-grid (see Section 5.2). The 2-D mode equations contain the three unknown variables U, V and ζ. The surface elevation is obtained from the continuity equation which does not explicitly require knowledge of ζ at the open boundaries. This means that open boundary conditions only need to be supplied for the transports U and V. However, a robust scheme needs to take account of the information entering or leaving the domain which involves all three parameters and should therefore include ζ as well.

The implemented schemes can be divided in four categories[11]:

1. Conditions without externally imposed values for transports and elevations (0,1,2,5,6,7,10,13).
2. Conditions with imposed elevations (3,9,12). An "external" value for the transport is obtained by solving a local solution of the momentum equations.
3. Conditions with imposed transports (4).
4. Conditions with specified transports and elevations (8,11).

External values (U^e, V^e, ζ^e) are expressed as the sum of a non-harmonic and an harmonic part

$$\psi^e(\xi_1,\xi_2,t) = \psi^e_0(\xi_1,\xi_2,t) + \sum_{n=1}^{N} A_n(\xi_1,\xi_2)f_n(t)\cos\Big(V_n(t)+u_n(t)-\varphi_n(\xi_1,\xi_2)\Big) \tag{4.354}$$

where ψ^e_0 represents the non-harmonic part, f_n, u_n are the nodal amplitude and phase factors, $V_n(t)$ the astronomical phases at Greenwich and A_n, φ_n the space-dependent amplitudes and phases with respect to Greenwich at

[11]The numbers in parentheses refer to the numbering of the descriptions below.

the open boundaries. The Greenwich phases and nodal factors are space-independent and obtained as function of time using the theory discussed in Section 4.5. The amplitudes A_n and phases φ_n are constant in time but non-uniform in space and needed to determine the harmonic input at the open boundaries. The function ψ_0^e depends on space and time. Values for A_n, φ_n and ψ_0^e need to be supplied by the user.

Variations in atmospheric pressure cause a displacement of the sea level. On the other hand, when an external surface elevation is specified, usually in the form of an harmonic expansion, the harmonic amplitudes are obtained with respect to a reference atmospheric pressure P_{ref}. To take account of this "inverse barometer" an (optionally) correction term is added to ζ^e, i.e.

$$\zeta^e = \text{expression (4.354)} \ + (P_{ref} - P_a)/(g\rho_0) \tag{4.355}$$

A relaxation condition can (optionally) be applied for all exterior 2-D data (transports and elevation) in case the model is set up with the default initial conditions (zero transports and elevations). In that case the exterior data function $\psi^e(\xi_1, \xi_2, t)$ is multiplied by the factor

$$\alpha_r(t) = \min((t - t_0)/T_r, 1) \tag{4.356}$$

where T_r is the relaxation period and t_0 the initial time. The method avoids the development of discontinuities during the initial propagation of (e.g.) a tidal wave into the domain.

All available schemes for 2-D open boundary conditions are briefly described below. Details are not given but can be found in the appropriate references. Comparison of different schemes are discussed in e.g. Palma & Matano (1998); Jensen (1998); Røed & Cooper (1987). Note that the conditions are applied after solving the 2-D mode equations for U, V, ζ at all interior points.

The following notations are adopted

- $\pm$ or $\mp$: upper (lower) sign applies at western/southern (eastern/northern) boundaries
- the gravity wave speed c is defined by $c = \sqrt{gH}$
- s equals 1 if ζ^e is defined at an exterior node or 2 if ζ^e is defined at the open boundary (U- or V-) node

0. Clamped.

   The transports are uniform in time and determined by the initial conditions.

$$\frac{\partial U}{\partial t} = 0 \ , \quad \frac{\partial V}{\partial t} = 0 \tag{4.357}$$

This is the default condition.

1. Zero slope.

   The 2-D momentum equations are solved without surface slope, advection, horizontal diffusion, pressure gradient and astronomical force.

$$\frac{\partial U}{\partial t} = fV + HF_1^t + \tau_{s1} - \tau_{b1} \ , \quad \frac{\partial V}{\partial t} = -fU + HF_2^t + \tau_{s2} - \tau_{b2} \quad (4.358)$$

2. Zero volume flux.

   This is a reflective boundary condition whereby the transport is set equal to its nearest interior value.

$$\frac{\partial U}{\partial \xi_1} = 0 \ , \quad \frac{\partial V}{\partial \xi_2} = 0 \quad (4.359)$$

3. Specified elevation.

   The 2-D momentum equations are solved without advection, horizontal diffusion, atmospheric and baroclinic pressure gradient.

$$\begin{aligned} \frac{\partial U}{\partial t} &= -\frac{c^2}{h_1}\frac{\partial \zeta}{\partial \xi_1} + fV + HF_1^t + \tau_{s1} - \tau_{b1} \\ \frac{\partial V}{\partial t} &= -\frac{c^2}{h_2}\frac{\partial \zeta}{\partial \xi_2} - fU + HF_2^t + \tau_{s2} - \tau_{b2} \end{aligned} \quad (4.360)$$

   The slope term is calculated by the spatial difference of the specified ζ value, either at the open boundary or outside the model grid, and its nearest interior value. The solutions are called "local" since all other horizontal gradient are suppressed. This condition is the easiest to use if ζ is the only available data parameter.

4. Specified transport.

$$U = U^e \ , \quad V = V^e \quad (4.361)$$

   This is the simplest and most appropriate condition to be used at river boundaries.

5. Radiation condition using the shallow water wave speed.

   Several types of radiation conditions, which allow the propagation of waves approaching the open boundary, are implemented. Oblique waves are not considered so that a normal incidence on the boundary is assumed. The methods use a Sommerfeld type of equation of the form

$$\frac{\partial \psi}{\partial t} \mp \frac{C}{h_i}\frac{\partial \psi}{\partial \xi_i} = 0 \quad (4.362)$$

where C is an appropriate wave speed. The first condition uses the surface gravity wave speed for shallow water

$$\frac{\partial U}{\partial t} \mp \frac{c}{h_1}\frac{\partial U}{\partial \xi_1} = 0 \; , \quad \frac{\partial V}{\partial t} \mp \frac{c}{h_2}\frac{\partial V}{\partial \xi_2} = 0 \tag{4.363}$$

6. Orlanski (1976) condition.

   This is the most popular radiation scheme using

$$C = c_r = h_i \frac{\partial\psi/\partial t}{\partial\psi/\partial\xi_i} \tag{4.364}$$

   so that

$$\begin{aligned} \frac{\partial U}{\partial t} \mp \frac{c_r}{h_1}\frac{\partial U}{\partial \xi_1} &= 0 \; , \quad c_r = \pm\frac{\partial U}{\partial t}\Big/\left(\frac{1}{h_1}\frac{\partial U}{\partial \xi_1}\right) \\ \frac{\partial V}{\partial t} \mp \frac{c_r}{h_2}\frac{\partial V}{\partial \xi_2} &= 0 \; , \quad c_r = \pm\frac{\partial V}{\partial t}\Big/\left(\frac{1}{h_2}\frac{\partial V}{\partial \xi_2}\right) \end{aligned} \tag{4.365}$$

   The numerical implementation (further discussed in Section 5.3.16.1) involves the values from two previous time steps and at the nearest two interior grid points.

7. Camerlengo & O'Brien (1980).

   The scheme is a mixture of the clamped and zero flux condition and can be considered as a simplified case of the Orlanski condition. Details are given in Section 5.3.16.1.

8. Flather (1976) with specified elevation and transport.

   This is based on (4.363) combined with the continuity equation using only the volume flux normal to the open boundary. The condition then requires that $U \pm c\zeta$ or $V \pm c\zeta$ is continuous across the boundary or

$$U = U^e \mp \frac{1}{2}sc(\zeta - \zeta^e) \; , \quad V = V^e \mp \frac{1}{2}sc(\zeta - \zeta^e) \tag{4.366}$$

9. Flather (1976) with specified elevation.

   The formulation is the same as (4.366) with U^e, V^e replaced by the local solutions U^L, V^L obtained by solving (4.360)

$$U = U^L \mp \frac{1}{2}sc(\zeta - \zeta^e) \; , \quad V = V^L \mp \frac{1}{2}sc(\zeta - \zeta^e) \tag{4.367}$$

10. Røed & Smedstad (1984).

    The condition is the same as Flather's condition but with all specified exterior values replaced by local solutions

$$U = U^L \mp c(\zeta - \zeta^L)\ , \quad V = V^L \mp c(\zeta - \zeta^L) \tag{4.368}$$

    where

$$\frac{\partial \zeta^L}{\partial t} = -\frac{1}{h_1 h_2}\frac{\partial}{\partial \xi_2}(h_1 V)\ , \quad \frac{\partial \zeta^L}{\partial t} = -\frac{1}{h_1 h_2}\frac{\partial}{\partial \xi_1}(h_2 U) \tag{4.369}$$

    and U^L, V^L are obtained from (4.360) using ζ^L.

11. Characteristic method with specified elevation and transport.

    The method is perhaps the most robust, but also the most complex one. The scheme is based on the theory of characteristics. The principle is to determine which information propagates into or out of the domain. The former is calculated using external data, the latter from the model equations. The method is discussed in Hedstrom (1979); Hirsch (1990) and applied in modified form to a barotropic ocean model by Røed & Cooper (1987).

    The characteristic variables are defined by

$$R^u_\pm = U \pm c\zeta \quad \text{or} \quad R^v_\pm = V \pm c\zeta \tag{4.370}$$

    At a western (eastern) boundary R^u_- (R^u_+) is the outgoing and R^u_+ (R^u_-) the incoming characteristic. Let $R^u_i = U \pm c\zeta$, $R^u_o = U \mp c\zeta$ denote the incoming and outgoing characteristics at a western or eastern boundary and $R^v_i = V \pm c\zeta$, $R^v_o = V \mp c\zeta$ their counterparts at a southern or northern boundary. The outgoing characteristics are obtained by solving

$$\frac{\partial R^u_o}{\partial t} \mp \frac{c}{h_1}\frac{\partial R^u_o}{\partial \xi_1} = \pm\frac{c}{h_1 h_2}\left(\frac{\partial}{\partial \xi_2}(h_1 V) + \frac{\partial h_2}{\partial \xi_1}U\right) + fV + HF^t_1 + \tau_{s1} - \tau_{b1} \tag{4.371}$$

    and

$$\frac{\partial R^v_o}{\partial t} \mp \frac{c}{h_2}\frac{\partial R^v_o}{\partial \xi_2} = \pm\frac{c}{h_1 h_2}\left(\frac{\partial}{\partial \xi_1}(h_2 U) + \frac{\partial h_1}{\partial \xi_2}V\right) - fU + HF^t_2 + \tau_{s2} - \tau_{b2} \tag{4.372}$$

    The equations are obtained by adding the continuity equation (4.85), multiplied by $\mp c$, to the two momentum equations (4.86)–(4.87), where (for convenience) the advective and horizontal diffusion terms, the baroclinic and atmospheric pressure gradient are neglected. These equations are solved using values of the involved parameters evaluated inside the domain.

The incoming characteristic is prescribed by

$$R_i^u = U^e \pm c\zeta^e \ , \quad R_i^v = V^e \pm c\zeta^e \tag{4.373}$$

The transports are then obtained by

$$U = \frac{1}{2}(R_i^u + R_o^u) \ , \quad V = \frac{1}{2}(R_i^v + R_o^v) \tag{4.374}$$

12. Characteristic method with specified elevation.

    The method is as previous with U^e, V^e replaced by the local solution U^L, V^L from (4.360).

13. Characteristic method using a zero normal gradient.

    Following Røed & Cooper (1987) the incoming characteristic is, in the absence of available data, obtained from (4.371) or (4.372) with $\partial R_i^u/\partial\xi_1 = 0$ or $\partial R_i^v/\partial\xi_2 = 0$. This gives

$$\frac{\partial R_i^u}{\partial t} = \mp\frac{c}{h_1h_2}\left(\frac{\partial}{\partial\xi_2}(h_1V) + \frac{\partial h_2}{\partial\xi_1}U\right) + fV + HF_1^t + \tau_{s1} - \tau_{b1} \tag{4.375}$$

$$\frac{\partial R_i^v}{\partial t} = \mp\frac{c}{h_1h_2}\left(\frac{\partial}{\partial\xi_1}(h_2U) + \frac{\partial h_1}{\partial\xi_2}V\right) - fU + HF_2^t + \tau_{s2} - \tau_{b2} \tag{4.376}$$

4.10.2 Open boundary conditions for the 3-D mode

4.10.2.1 baroclinic currents

Since U and V are obtained from the 2-D open boundary conditions, only their baroclinic parts $\delta u = U - u/H$ and $\delta v = V - v/H$ need to be determined. The following conditions can be selected

0. Zero gradient.

$$\frac{1}{h_1}\frac{\partial}{\partial\xi_1}(h_2h_3\delta u) = 0 \ , \quad \frac{1}{h_2}\frac{\partial}{\partial\xi_2}(h_1h_3\delta v) = 0 \tag{4.377}$$

   This is the default condition.

1. Specified external profile.

$$\delta u = \delta u^e \; , \quad \delta v = \delta v^e \tag{4.378}$$

2. Second order gradient condition. In case of ragged open boundaries the (first order) zero gradient condition may yield spurious discontinuities of the vertical current at the first interior node. The effect is reduced when using the second order condition

$$\frac{1}{h_1}\frac{\partial}{\partial \xi_1}\Big[\frac{1}{h_1 h_2}\frac{\partial}{\partial \xi_1}(h_2 h_3 \delta u)\Big] = 0 \, , \quad \frac{1}{h_2}\frac{\partial}{\partial \xi_2}\Big[\frac{1}{h_1 h_2}\frac{\partial}{\partial \xi_2}(h_1 h_3 \delta v)\Big] = 0 \tag{4.379}$$

at respectively U- and V-node open boundaries.

3. Local solution. The equation is derived from the 3-D and 2-D momentum equations without advection and horizontal diffusion:

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 \delta u) - 2\Omega \sin\phi \delta v = F_1^b - \frac{\overline{F_1^b}}{H} + \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\nu_T}{h_3}\frac{\partial \delta u}{\partial s}\Big) + \frac{\tau_{b1} - \tau_{s1}}{H} \tag{4.380}$$

at U-nodes and

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 \delta v) + 2\Omega \sin\phi \delta u = F_2^b - \frac{\overline{F_2^b}}{H} + \frac{1}{h_3}\frac{\partial}{\partial s}\Big(\frac{\nu_T}{h_3}\frac{\partial \delta v}{\partial s}\Big) + \frac{\tau_{b2} - \tau_{s2}}{H} \tag{4.381}$$

at V-node open boundaries. The diffusive fluxes at the surface and bottom are determined by respectively (4.265) and (4.336) with u, v, replaced by δu, δv.

4. Radiation condition using the baroclinic wave speed.

$$\frac{\partial \delta u}{\partial t} \mp c_i \frac{1}{h_1}\frac{\partial \delta u}{\partial \xi_1} = 0 \, , \quad \frac{\partial \delta v}{\partial t} \mp c_i \frac{1}{h_2}\frac{\partial \delta v}{\partial \xi_2} = 0 \tag{4.382}$$

The baroclinic wave speed c_i is generally unknown and has to be specified by the user.

5. Orlanski condition.

$$\begin{aligned} \frac{\partial \delta u}{\partial t} \mp c_r \frac{1}{h_1}\frac{\partial \delta u}{\partial \xi_1} &= 0 \, , \quad c_r = \pm \frac{\partial \delta u}{\partial t} \Big/ \Big(h_1 \frac{\partial \delta u}{\partial \xi_1}\Big) \\ \frac{\partial \delta v}{\partial t} \mp c_r \frac{1}{h_2}\frac{\partial \delta v}{\partial \xi_2} &= 0 \, , \quad c_r = \pm \frac{\partial \delta v}{\partial t} \Big/ \Big(h_2 \frac{\partial \delta v}{\partial \xi_2}\Big) \end{aligned} \tag{4.383}$$

4.10.2.2 3-D scalars

Open boundary conditions are needed for calculation of the horizontal advective fluxes inside the domain.

0. Zero gradient.

$$\frac{1}{h_1}\frac{\partial}{\partial \xi_1}(h_2 h_3 \psi) = 0 \ , \quad \frac{1}{h_2}\frac{\partial}{\partial \xi_2}(h_1 h_3 \psi) = 0 \tag{4.384}$$

   This is the default condition.

1. Specified external profile.

$$\psi = \psi^e \tag{4.385}$$

2. Radiation condition using the baroclinic wave speed.

$$\frac{\partial \psi}{\partial t} \mp c_i \frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1} = 0 \ , \quad \frac{\partial \psi}{\partial t} \mp c_i \frac{1}{h_2}\frac{\partial \psi}{\partial \xi_2} = 0 \tag{4.386}$$

   The baroclinic wave speed c_i is generally unknown and has to be specified by the user.

3. Orlanski condition.

$$\begin{aligned} \frac{\partial \psi}{\partial t} \mp c_r \frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1} &= 0 \ , \quad c_r = \pm \frac{\partial \psi}{\partial t} / \left(h_1 \frac{\partial \psi}{\partial \xi_1} \right) \\ \frac{\partial \psi}{\partial t} \mp c_r \frac{1}{h_2}\frac{\partial \psi}{\partial \xi_2} &= 0 \ , \quad c_r = \pm \frac{\partial \psi}{\partial t} / \left(h_2 \frac{\partial \psi}{\partial \xi_2} \right) \end{aligned} \tag{4.387}$$

4.10.2.3 turbulence variables

Advection of turbulence is considered of minor importance and has been disabled by default. The only available open boundary condition for k, ε or kl therefore consists of a zero gradient condition.

4.10.3 Relaxation conditions

A known problem with open boundary conditions is that the imposed value of a quantity at the open boundary is not always compatible with its value calculated by the model inside. For example, the thermocline depth obtained from an imposed vertical profile of temperature may be different from the one calculated by the model just inside the domain, creating unrealistic discontinuities. A known solution is the creation of a sponge layer near the open boundaries where the calculated interior solution is allowed to relax towards its value imposed at the open boundary.

The method implemented in the program is the flow relaxation scheme proposed by Martinsen & Engedahl (1987). A relaxation zone is defined near the open boundary where the value of a model quantity ψ is written as

$$\psi_i = \alpha\psi_e + (1-\alpha)\tilde{\psi}_i \tag{4.388}$$

where ψ_e is the open boundary value, $\tilde{\psi}_i$ the calculated interior value, prior to relaxation, and α a weighting factor which varies between 1 at the open boundary and 0 at the inner edge of the relaxation zone. Martinsen & Engedahl (1987) showed for a simplified case that the procedure is equivalent to add a relaxation term $\alpha(\psi-\psi_e)/\Delta t/(1-\alpha)$ to the right hand side of the transport equation where Δt is the model time step. In this way, ψ relaxes to its open boundary value if $\alpha \to 1$ and to the internal solution if $\alpha \to 0$.

The method can be applied in the program for temperature, salinity and baroclinic currents, but is not implemented for the 2-D mode. The following interpolation schemes can be selected

1. linear
$$\alpha = 1 - \frac{d}{D} \tag{4.389}$$

2. quadratic
$$\alpha = \Big(1 - \frac{d}{D}\Big)^2 \tag{4.390}$$

3. hyperbolic
$$\alpha = 1 - \tanh\Big(\frac{d}{2\Delta h}\Big) \tag{4.391}$$

   where d is the distance to the boundary, D the width of the relaxation zone and Δh the grid spacing. Note that the interpolation assumes a uniform grid spacing within the relaxation zone.

4.10.4 Coastal boundaries

At coastal boundaries currents and fluxes of of scalars are set to zero, or

$$U = 0\,, \quad \delta u = 0\,, \quad u\psi = 0 \tag{4.392}$$

at western and eastern boundaries, and

$$V = 0\,, \quad \delta v = 0\,, \quad v\psi = 0 \tag{4.393}$$

at southern and northern boundaries.

4.11 Initial conditions

The default initial conditions are

- 2-D hydrodynamics

$$U = 0, \qquad V = 0, \qquad \zeta = 0 \tag{4.394}$$

- 3-D hydrodynamics

$$u = v = 0 \tag{4.395}$$

- scalars

$$T = T_{ref}, \qquad S = S_{ref} \tag{4.396}$$

  where T_{ref}, S_{ref} are uniform values selected by the user. These are also the values taken over time if iopt_temp=0 or iopt_sal=0.

- turbulence

$$k = 10^{-6}\,\mathrm{J/kg} \tag{4.397}$$

  for turbulent energy while l is obtained from one of the mixing length prescriptions given in Section 4.4.3.5 and ε from (4.203).

Although the initial conditions for turbulence cannot be considered as realistic, they are of lesser importance since turbulence is assumed to be in quasi-equilibrium and adjusts itself rapidly to changes in the forcing conditions, given by the vertical current shear and stratification. It is clear that realistic initial conditions cannot be given in general. In practice, they need to be obtained by the user.

4.12 Harmonic analysis

4.12.1 Residuals, amplitudes and phases

The program offers the possibility to apply an harmonic analysis on a given number of user-defined quantities. The method is closely related to the one described in Godin (1972). An harmonic expansion approximates a function $F(\xi_1, \xi_2, \sigma, t)$ by a series of the form

$$F(\xi_1, \xi_2, \sigma, t) = a_0 + \sum_{n=1}^{N_h} a_n \cos\omega_n(t - t_c) + \sum_{n=1}^{N_h} b_n \sin\omega_n(t - t_c) \tag{4.398}$$

where a_0, a_n and b_n are spatially dependent parameters obtained with an optimisation procedure, ω_n are a series of user-defined frequencies, N_h the number of harmonics in the analysis, t_c a "central" time and t the time since the start of the simulation. The procedure uses a least-squares fitting. Firstly, the time t_c and a period T for the analysis, given by an even number of time steps[12], i.e. $T = 2M\Delta t$, are defined. Secondly, the program evaluates the values of F at times $t_c + k\Delta t$ with $-M \leq k \leq M$, denoted by

$$F_k = F(\xi_1, \xi_2, \sigma, t_c + k\Delta t) \tag{4.399}$$

The harmonic parameters a_0, a_n and b_n are then determined by minimising the quantity

$$R = \sum_{k=-M}^{M} [F_k - a_0 - \sum_{n=1}^{N_h} a_n \cos(\omega_n k\Delta t) - \sum_{n=1}^{N_h} b_n \sin(\omega_n k\Delta t)]^2 \tag{4.400}$$

Setting the first derivatives of R with respect to a_0, a_m and b_m ($1 \leq m \leq N_h$) equal to zero yields the following set of $2N_h + 1$ equations

$$\sum_{k=-M}^{M} [F_k - a_0 - \sum_{n=1}^{N_h} a_n \cos(\omega_n k\Delta t) - \sum_{n=1}^{N_h} b_n \sin(\omega_n k\Delta t)] = 0 \tag{4.401}$$

$$\sum_{k=-M}^{M} [F_k - a_0 - \sum_{n=1}^{N_h} a_n \cos(\omega_n k\Delta t) - \sum_{n=1}^{N_h} b_n \sin(\omega_n k\Delta t)] \cos(\omega_m k\Delta t) = 0 \tag{4.402}$$

$$\sum_{k=-M}^{M} [F_k - a_0 - \sum_{n=1}^{N_h} a_n \cos(\omega_n k\Delta t) - \sum_{n=1}^{N_h} b_n \sin(\omega_n k\Delta t)] \sin(\omega_m k\Delta t) = 0 \tag{4.403}$$

with $1 \leq m \leq N_h$. The system can be simplified with the aid of known summation rules for trigonometric functions (see Gradshteyn & Ryzhik, 1981). The final result can be written as

$$\sum_{n=1}^{N_h} X_{mn} a_n = R_m \tag{4.404}$$

$$\sum_{n=1}^{N_h} Y_{mn} b_n = S_m \tag{4.405}$$

$$a_0 = \frac{1}{2M+1}\Big[\sum_{k=-M}^{M} F_k - \sum_{n=1}^{N_h} a_n \sin(\frac{2M+1}{2}\omega_n \Delta t) \csc(\frac{\omega_n \Delta t}{2})\Big] \tag{4.406}$$

[12]Note that Δt equals the time step for the 3-D mode calculations.

where

$$
\begin{aligned}
X_{mn} = & \frac{1}{2}\sin\Big(\frac{2M+1}{2}(\omega_m+\omega_n)\Delta t\Big)\csc\Big(\frac{\Delta t}{2}(\omega_m+\omega_n)\Big) \\
& +\frac{1}{2}\sin\Big(\frac{2M+1}{2}(\omega_m-\omega_n)\Delta t)\csc\Big(\frac{\Delta t}{2}(\omega_m-\omega_n)\Big) \\
& -\frac{1}{2M+1}\sin(\frac{2M+1}{2}\omega_m\Delta t)\csc(\frac{\omega_m\Delta t}{2})\sin(\frac{2M+1}{2}\omega_n\Delta t)\csc(\frac{\omega_n\Delta t}{2})
\end{aligned}
\tag{4.407}
$$

$$
\begin{aligned}
Y_{mn} = & \frac{1}{2}\sin\Big(\frac{2M+1}{2}(\omega_m-\omega_n)\Delta t\Big)\csc\Big(\frac{\Delta t}{2}(\omega_m-\omega_n)\Big) \\
& -\frac{1}{2}\sin\Big(\frac{2M+1}{2}(\omega_m+\omega_n)\Delta t\Big)\csc\Big(\frac{\Delta t}{2}(\omega_m+\omega_n)\Big)
\end{aligned}
\tag{4.408}
$$

$$
R_m = \sum_{k=-M}^{M} F_k\cos(\omega_m k\Delta t) - \frac{1}{2M+1}\sin(\frac{2M+1}{2}\omega_m\Delta t)\csc(\frac{\omega_m\Delta t}{2})\sum_{k=-M}^{M} F_k \tag{4.409}
$$

$$
S_m = \sum_{k=-M}^{M} F_k\sin(\omega_m k\Delta t) \tag{4.410}
$$

The following replacements need to be made at singular points of the csc (cosecans) functions:

$$
\sin\Big(\frac{2M+1}{2}\omega_*\Delta t\Big)\csc\Big(\frac{\omega_*\Delta t}{2}\Big) \to 2M+1 \quad \text{if} \quad \mathrm{mod}(\omega_*\Delta t, 2\pi) = 0 \tag{4.411}
$$

where ω_* equals either ω_m , ω_n , $\omega_m+\omega_n$, $\omega_m-\omega_n$. The matrices X and Y are symmetric and depend only on the values of the frequencies, the time step and the analysed period and can thus be evaluated at the start of the program. The numerical solution of the linear system (4.404) and (4.405) is facilitated by first performing a Cholesky decomposition on the matrices X and Y, which only needs to be performed once. This reduces the number of arithmetic operations and computing time since the systems are to be solved at a number of selected grid points and for a given number of user-defined quantities. Details of the numerical procedure are described in Press *et al.* (1992).

Once the parameters a_0, a_n and b_n are determined, the harmonic expansion (4.398) is written into the form

$$
F(\xi_1,\xi_2,\sigma,t) = A_0 + \sum_{n=1}^{N_h} A_n\cos(\omega_n(t-t_c)-\varphi_{nc}) \tag{4.412}
$$

where the residual A_0, the amplitudes A_n and the phases φ_{nc} (with respect to the central time t_c) are obtained from

$$A_0 = a_0 \ , \ A_n = (a_n^2 + b_n^2)^{1/2} \ , \ \varphi_{nc} = \mathrm{mod}(\arctan(b_n/a_n), 2\pi) \qquad (4.413)$$

The actual phase φ_n can be obtained in the program with respect to either

1. the central time

$$\varphi_n = \varphi_{nc} \qquad (4.414)$$

2. the astronomical Greenwich phase. Letting

$$\varphi_n = \varphi_{nc} + V_n(t_c) + u_n(t_c) \qquad (4.415)$$

One has

$$\omega_n(t - t_c) - \varphi_{nc} \simeq V_n(t) + u_n(t) - \varphi_n \qquad (4.416)$$

3. a given reference date defined by

$$t_{ref} = t + \Delta_{ref} \qquad (4.417)$$

where Δ_{ref} is time difference between the initial date of the simulation and the reference date. Letting

$$\varphi_n = \varphi_{nc} - \omega_n(t_c + \Delta_{ref}) \qquad (4.418)$$

one obtains

$$\omega_n(t - t_c) - \varphi_{nc} = \omega_n t_{ref} - \varphi_n \qquad (4.419)$$

Important to note is that the number and values of the frequencies ω_n used in the harmonic analysis do not need to be the same as the ones appearing in the tidal forcing, as given by the harmonic expansions (4.230) or (4.354). For example, if the model is forced with the M_2-tide only, higher order harmonics (M_4, M_6, ...) are generated by the non-linearities of the model equations. These higher order terms can be investigated by applying an harmonic analysis. The method can be also used to analyse the evolution of a M_2-tide during a spring-neaps cycle, e.g. by forcing the model with a M_2- and S_2-tide and performing the analysis with only the M_2-frequency at different times $t = t_c, t_c + T, \ldots$ (e.g. Luyten, 1997). Luyten *et al.* (2003) applied the same procedure for the analysis of internal waves in the North Sea.

The period T needs to be selected with some care. A general rule is that it must be of the same order or larger than any of the analysed periods $2\pi/\omega_n$ and must increase with the number of frequencies. A more stringent restriction is imposed if two freqiencies ω_i and ω_j are nearly equal to avoid aliasing effects, in which case T should larger than $2\pi/|\omega_i - \omega_j|$.

4.12.2 Tidal ellipses

During the course of a tidal cycle the current vector describes a curve, known as the "tidal ellipse". Characteristic parameters of tidal ellipses are the semi-major axis, semi-minor axis, ellipticity, orientation and elliptic angle. They can optionally be derived by the program in the usual way (e.g. Godin, 1972) once the harmonic parameters of the current are available using the analysis of the preceding section. The "nth" harmonic component of the horizontal current can be written as

$$u_n = u_{an}\cos(\omega_n t - \varphi_{nu}) \ , \ v_n = v_{an}\cos(\omega_n t - \varphi_{nv}) \tag{4.420}$$

Introducing the complex notation

$$U = u_a e^{-i\varphi_u} \ , \ V = v_a e^{-i\varphi_v} \tag{4.421}$$

where the subscript n has been omitted for simplicity, the complex current can then be decomposed into a cyclonically and an anticyclonically rotating component

$$\begin{aligned} u + iv &= \frac{1}{2}(Ue^{i\omega t} + \tilde{U}e^{-i\omega t}) + \frac{i}{2}(Ve^{i\omega t} + \tilde{V}e^{-i\omega t}) \\ &= S_+e^{i\omega t} + \tilde{S}_-e^{-i\omega t} \end{aligned} \tag{4.422}$$

where a ˜ denotes the complex conjugate and

$$S_\pm = \frac{1}{2}(U \pm iV) = |S_\pm|e^{i\alpha_\pm} \tag{4.423}$$

The semi-major axis A, semi-minor axis B and ellipticity e of the ellipse, described by the current, are then given by

$$A = |S_+| + |S_-| \ , \ B = ||S_+| - |S_-|| \ , \ e = (|S_+| - |S_-|)/(|S_+| + |S_-|) \tag{4.424}$$

The inclination Θ of the ellipse with respect to the ξ_1-axis and the elliptic angle Φ, i.e. the angle between the initial current at $t = 0$ and its position when the current achieves its first maximum, are obtained using

$$\Theta = (\alpha_+ - \alpha_-)/2 \ , \ \Phi = -(\alpha_+ + \alpha_-)/2 \tag{4.425}$$

with the restriction that

$$0 \leq \Theta, \Phi \leq \pi \tag{4.426}$$

The orientation of the ellipse is determined by the sign of the ellipticity. A positive value of e means that the current vector rotates cyclonically (anti-clockwise in the northern hemisphere) and $|S_+| > |S_-|$ whereas a negative value indicates anticyclonic rotation (clockwise in the northern hemisphere) and $|S_+| < |S_-|$. If $e = 0$, the flow is rectilinear and $|S_+| = |S_-|$. A useful discussion of cyclonic and anticyclonic components and their impact on the depth of the tidal bottom layer can be found in e.g. Prandle (1982); Soulsby (1983); Luyten (1996).

Implementation

The following switches are available for harmonic analysis

- iopt_out_anal — Switch to enable (1) or disable (0) harmonic analysis
- iopt_astro_anal — If iopt_astro_anal=1 and cdate_timeref is not defined, the harmonic phases φ_n are calculated with respect to the astronomical phase at Greenwich. Otherwise the phases are obtained with respect to a given reference date or to the central time, depending on whether cdate_timeref is defined or not (see Section 20.3.1).