Appendix D

Energy balance equation

The energy balance equation is derived by multiplying equations (4.61), (4.62), (4.60) by respectively $\rho_0 u$, $\rho_0 v$, $\rho_0 g\zeta$ and adding. For simplicity the baroclinic and atmospheric pressure gradients, the tidal force and horizontal diffusion are neglected. After a straightforward calculation one obtains

$$\frac{1}{h_3}\frac{\partial}{\partial t}(h_3 E_k) + \frac{1}{H}\frac{\partial E_p}{\partial t} + \frac{1}{h_1 h_2 h_3}\Big[\frac{\partial}{\partial \xi_1}(h_2 h_3 F_1) + \frac{\partial}{\partial \xi_2}(h_1 h_3 F_2)\Big] = D \quad \text{(D.1)}$$

where

$$E_k = \frac{1}{2}\rho_0(u^2 + v^2) \quad \text{(D.2)}$$

is the kinetic energy,

$$E_p = \frac{1}{2}\rho_0 g\zeta^2 \quad \text{(D.3)}$$

the potential energy,

$$(F_1, F_2) = \Big(\frac{1}{2}\rho_0(u^2 + v^2) + \rho_0 g\zeta\Big)(u, v, \omega) \quad \text{(D.4)}$$

the energy flux vector and

$$D = \rho_0\Big(\frac{u}{h_3}\frac{\partial D_1}{\partial s} + \frac{v}{h_3}\frac{\partial D_2}{\partial s}\Big) \quad \text{(D.5)}$$

the dissipation of energy by turbulent diffusion. The diffusion fluxes are given by

$$\begin{aligned}(D_1, D_2) &= \Big(\frac{\nu_T}{h_3}\frac{\partial u}{\partial s}, \frac{\nu_T}{h_3}\frac{\partial v}{\partial s}\Big) && \text{inside the water column}\\ &= (\tau_{s1}, \tau_{s2}) && \text{at the surface}\\ &= (\tau_{b1}, \tau_{b2}) && \text{at the bottom}\end{aligned} \quad \text{(D.6)}$$

The vertically integrated form of (D.1) becomes

$$\frac{\partial}{\partial t}(\overline{E}_k + E_p) + \frac{1}{h_1 h_2}\Big[\frac{\partial}{\partial \xi_1}(h_2 \overline{F}_1) + \frac{\partial}{\partial \xi_2}(h_1 \overline{F}_2)\Big] = \overline{D} \tag{D.7}$$

where

$$\overline{E}_k = \frac{1}{2}\rho_0 \int_0^{N_z} h_3(u^2+v^2)\,ds \tag{D.8}$$

$$(\overline{F_1}, \overline{F}_2) = \int_0^{N_z} h_3\Big[\frac{1}{2}\rho_0(u^2+v^2) + \rho_0 g\zeta\Big](u,v)\,ds \tag{D.9}$$

$$\overline{D} = \rho_0(u_s\tau_{s1} + v_s\tau_{s2} - u_b\tau_{b1} - v_b\tau_{b2}) - \rho_0 \int_0^{N_z} \frac{\nu_T}{h_3}\Big[\Big(\frac{\partial u}{\partial s}\Big)^2 + \Big(\frac{\partial v}{\partial s}\Big)^2\Big]\,ds \tag{D.10}$$

where (u_s,v_s) and (u_b,v_b) are the velocities at respectively the surface and the bottom.

In case of a pure 2-D application, $u \simeq \overline{u}$ and $v \simeq \overline{v}$ in which case (D.7)–(D.10) reduce to

$$\overline{E}_k = \frac{1}{2}\rho_0 H(\overline{u}^2 + \overline{v}^2) \tag{D.11}$$

$$(\overline{F_1}, \overline{F}_2) = H\Big[\frac{1}{2}\rho_0(\overline{u}^2+\overline{v}^2) + \rho_0 g\zeta\Big](\overline{u},\overline{v}) \tag{D.12}$$

$$\overline{D} = \rho_0(u_s\tau_{s1} + v_s\tau_{s2} - u_b\tau_{b1} - v_b\tau_{b2}) \tag{D.13}$$

Finally, the domain integrated forms are obtained by integrating (D.7) over the 2-D horizontal domain. This gives

$$\frac{\partial}{\partial t}\Big(\langle\overline{E}_k\rangle + \langle E_p\rangle\Big) + \int(\overline{F}_1 n_1 + \overline{F}_2 n_2)\,d\ell = \langle\overline{D}\rangle \tag{D.14}$$

with

$$\langle\cdots\rangle = \int_0^{N_x}\int_0^{N_y}(\cdots)h_1 h_2\,d\xi_1 d\xi_2 \tag{D.15}$$

The second integral is taken along the 2-D boundary of the domain where (n_1,n_2) is the outwards pointing unit normal along the boundary.