Appendix B

Solutions of the RANS equations

B.1 Non-equilibrium method

Using the definitions (4.179) and (4.181), the RANS equations (4.178) can be reduced to two linear equations for the stability functions S_u and S_b. Firstly, the following auxiliary parameters are defined

$$
\begin{aligned}
\alpha_1 &= 1 - c_{21} - c_{23} + \frac{3}{2}c_{22} \\
\alpha_2 &= (1 - c_{21})^2 + c_{23}(4 - 4c_{21} + c_{23}) \\
\alpha_3 &= 3 - c_{21} + 2c_{23} - 2c_3 \\
\alpha_4 &= c_{23}(3 - 2c_{21} + c_{23} - c_3) \\
\alpha_5 &= \alpha_1 - \frac{\alpha_2}{c_1} \\
\alpha_6 &= (1 - c_{21})(3 - c_{21} + 2c_{23} - 2c_3) + \alpha_4 \\
R^\star_\beta &= R_\beta(1 - c_{3\beta})
\end{aligned}
\tag{B.1}
$$

The stability functions are then obtained as solution of

$$
\Big[c_1^2 c_{1\beta} + \frac{2}{3}c_{1\beta}\alpha_2\alpha_M + c_1(1 - c_3)\alpha_N\Big]S_u
+ (1 - c_3)\Big[\frac{2}{3}c_{1\beta}(2 - 2c_{21} + c_{23}) + c_1(1 - c_{21\beta})\Big]\alpha_N S_b = \frac{2}{3}c_1 c_{1\beta}\alpha_1
\tag{B.2}
$$

$$
\Big[\frac{2}{3}c_{1\beta}(1 - c_{21} + 2c_{23}) - c_{22\beta}c_1\Big]\alpha_M S_u \quad + \quad \Big[c_1 c_{1\beta}^2 - c_1 c_{22\beta}(1 - c_{21\beta})\alpha_M
$$

$$+ \quad 2c_{1\beta}\Big(c_1 R^{\star}_{\beta} + \frac{2}{3}(1-c_3)\Big)\alpha_N\Big] S_b = \frac{2}{3}c_1 c_{1\beta} \tag{B.3}$$

The solution can be written in the form, given by (4.182). The coefficients C_{a1}–C_{a11} take the following values

$$\begin{aligned}
C_{a1} &= \frac{2\alpha_1}{3c_1} \\
C_{a2} &= -\frac{2\alpha_1(1-c_{21\beta})c_{22\beta}}{3c_1 c_{1\beta}^2} \\
C_{a3} &= -\frac{2}{3c_1 c_{1\beta}}\Big[(1-c_3)\Big(\frac{2(c_{23}-c_{22})}{c_1} + \frac{1-c_{21\beta}}{c_{1\beta}}\Big) - 2R^{\star}_{\beta}\alpha_1\Big] \\
C_{a4} &= \frac{2}{3c_{1\beta}} \\
C_{a5} &= \frac{2}{3c_1 c_{1\beta}}\Big[\frac{2}{3c_1}\Big(\alpha_2 - \alpha_1(1-c_{21}+2c_{23})\Big) + \frac{c_{22\beta}}{c_{1\beta}}\alpha_1\Big] \\
C_{a6} &= \frac{2(1-c_3)}{3c_1 c_{1\beta}^2} \\
C_{a7} &= \frac{2\alpha_2}{3c_1^2} - \frac{c_{22\beta}(1-c_{21\beta})}{c_{1\beta}^2} \\
C_{a8} &= \frac{1}{c_{1\beta}}\left[\frac{7(1-c_3)}{3c_1} + 2R^{\star}_{\beta}\right] \\
C_{a9} &= -\frac{2c_{22\beta}(1-c_{21\beta})\alpha_2}{3c_1^2 c_{1\beta}^2} \\
C_{a10} &= \frac{2}{3c_1^2 c_{1\beta}}\Big\{(1-c_3)\Big[\frac{2c_{23}}{c_1}(1-c_{21}) \\
&\quad + \frac{1}{c_{1\beta}}\Big(c_{22\beta}(2-2c_{21}+c_{23}) - (1-c_{21\beta})(1-c_{21}+2c_{23})\Big)\Big] + 2R^{\star}_{\beta}\alpha_2\Big\} \\
C_{a11} &= \frac{2(1-c_3)}{c_1 c_{1\beta}^2}\left[\frac{2(1-c_3)}{3c_1} + R^{\star}_{\beta}\right]
\end{aligned} \tag{B.4}$$

B.2 Quasi-equilibrium method

The governing equations for the algebraic RANS model are given by (4.186) supplemented by the last 7 equations of (4.178). In analogy with the previous case they can be reduced to the following equations for S_u and S_b.

$$\begin{aligned} c_1[c_1c_{1\beta} + (1-c_3)\alpha_N]S_u \quad &+ \quad \Big[\frac{2}{3}c_{1\beta}\alpha_6 + c_1(1-c_3)(1-c_{21\beta})\Big]\alpha_N S_b \\ &= \quad \frac{2}{3}c_1c_{1\beta}\alpha_5 \end{aligned} \tag{B.5}$$

$$\begin{aligned} \Big[c_1c_{1\beta}^2 - c_1(1-c_{21\beta})c_{22\beta}\alpha_M \quad &+ \quad (2c_1c_{1\beta}R_\beta^\star + \frac{2}{3}c_{1\beta}\alpha_3 - c_1c_{22\beta})\alpha_N\Big]S_b \\ &= \quad \frac{2}{3}c_{1\beta}(c_1 - 1 + c_{21} - 2c_{23}) + c_1c_{22\beta} \end{aligned} \tag{B.6}$$

If $c_{22\beta} = 0$, the solutions can be cast in the form (4.187) where

$$\begin{aligned} C_{b1} &= \frac{2\alpha_5}{3c_1} \\ C_{b2} &= \frac{2}{3c_1c_{1\beta}}\Big[\frac{2}{3c_1}(\alpha_3\alpha_5 - \alpha_6) + \frac{2\alpha_6}{3c_1^2}(1 - c_{21} + 2c_{23}) \\ &\quad - \frac{(1-c_3)(1-c_{21\beta})}{c_{1\beta}}\Big(1 - \frac{1}{c_1}(1 - c_{21} + 2c_{23})\Big) + 2\alpha_5R_\beta^\star\Big] \\ C_{b3} &= \frac{2}{c_{1\beta}}\Big(R_\beta^\star + \frac{\alpha_3}{3c_1}\Big) \\ C_{b4} &= \frac{1-c_3}{c_1c_{1\beta}} \\ C_{b5} &= \frac{2}{3c_{1\beta}}\left(1 - \frac{1}{c_1}(1 - c_{21} + 2c_{23})\right) \end{aligned} \tag{B.7}$$

If $c_{22\beta} \neq 0$ the solutions are given by (4.188)–(4.189) with

$$\begin{aligned} C_{c1} &= \frac{2\alpha_5}{3c_1} \\ C_{c2} &= -\frac{1}{c_1}\left[\frac{2\alpha_6}{3c_1} + \frac{(1-c_3)(1-c_{21\beta})}{c_{1\beta}}\right] \\ C_{c3} &= \frac{1-c_3}{c_1c_{1\beta}} \\ C_{c4} &= -\Big[\frac{2\alpha_6}{3c_1^2} + \frac{1-c_{21\beta}}{c_{1\beta}}\left(\frac{c_{22\beta}}{c_{1\beta}} + \frac{1-c_3}{c_1}\right)\Big] \\ C_{c5} &= -\frac{2}{3c_1c_{1\beta}}\Big[\frac{\alpha_6}{c_1}\left(2R_\beta^\star + \frac{2\alpha_3}{3c_1} - \frac{c_{22\beta}}{c_{1\beta}}\right) + \frac{(1-c_3)(1-c_{21\beta})}{c_{1\beta}}\left(3R_\beta^\star + \frac{\alpha_3}{c_1}\right)\Big] \\ C_{c6} &= \frac{2\alpha_5}{3c_1} - \frac{(1-c_{21\beta})c_{22\beta}}{c_{1\beta}^2} \end{aligned}$$

$$
\begin{aligned}
C_{c7} &= \frac{2}{3c_1c_{1\beta}}\Big[\frac{2}{3}\Big(\frac{\alpha_3\alpha_5+\alpha_6}{c_1} - \frac{\alpha_6(1-c_{21}+2c_{23})}{c_1^2}\Big) \\
&\quad + \frac{(1-c_3)(1-c_{21\beta})}{c_{1\beta}}\Big(1-\frac{1-c_{21}+2c_{23}}{c_1}\Big) - \frac{c_{22\beta}}{c_{1\beta}}(\alpha_5 - \frac{\alpha_6}{c_1}) + 2\alpha_5 R_\beta^\star\Big] \\
C_{c8} &= -\frac{2\alpha_5}{3c_1c_{1\beta}}\Big(\frac{2}{3} - \frac{2}{3c_1}(1-c_{21}+2c_{23}) + \frac{c_{22\beta}}{c_{1\beta}}\Big) \qquad \text{(B.8)}
\end{aligned}
$$

B.3 Equilibrium method

The coefficients in the quadratic equation (4.195) for κ^2 are obtained from equations (B.5)–(B.6) and the equilibrium relation $S_u\alpha_M - S_b\alpha_N = 1$. One has

$$
\begin{aligned}
C_{d1} &= -\frac{2\alpha_5}{3c_1} - \frac{(1-c_{21\beta})c_{22\beta}}{c_{1\beta}^2} \\
C_{d2} &= \frac{1}{c_{1\beta}}\Big(\frac{2}{3} + \frac{7}{3c_1}(1-c_3) + 2R_\beta^\star\Big) \\
C_{d3} &= \frac{2(1-c_{21\beta})c_{22\beta}\alpha_5}{3c_1c_{1\beta}^2} \\
C_{d4} &= \frac{2}{3c_1c_{1\beta}}\Big[\frac{2}{3c_1}\Big(\alpha_6 - \alpha_3\alpha_5 - \frac{(1-c_{21}+2c_{23})\alpha_6}{c_1}\Big) \\
&\quad + \frac{(1-c_3)(1-c_{21\beta})}{c_{1\beta}}\Big(1-\frac{1-c_{21}+2c_{23}}{c_1}\Big) + \frac{c_{22\beta}}{c_{1\beta}}\Big(\alpha_1 + \frac{1}{c_1}(1-c_3)(2-2c_{21}+c_{23})\Big) \\
&\quad - 2\alpha_5 R_\beta^\star\Big] \\
C_{d5} &= \frac{1-c_3}{c_1c_{1\beta}^2}\Big(\frac{2}{3} + \frac{4(1-c_3)}{3c_1} + 2R_\beta^\star\Big) \qquad \text{(B.9)}
\end{aligned}
$$